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Harmonic analysis of several complex variables: A survey

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Abstract

We treat basic issues of harmonic analysis in several complex variables. This includes the study of Hardy spaces, singular integrals, reproducing kernels, partial differential equations, and Fourier analysis. Along the way we indicate several new results in different aspects of the subject.

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1. Introduction

The history of harmonic analysis as we know it today goes back to the time of Leonhard Euler and even beyond. The basic ideas really gelled in the work of Jean Baptiste Joseph Fourier in the early nineteenth century. A fundamental feature of the subject is that harmonic analysts are always happy to seek new settings in which to ply their craft.

The harmonic analysis of several complex variables is also a relatively recent development. Some isolated results appeared in the 1940s and 1950s. But the subject did not really take off until about 1970. Since that time, there have been a number of seminal works that lay the foundations of the harmonic analysis of several complex variables. There still remains much work to be done, and that fact is a significant part of the motivation for this article. We hope to draw in a new group of workers to this fascinating topic.

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2. Exploratory works

In the papers [78,6], Caldéron and Zygmund began to explore the boundary behavior of holomorphic functions on domains in $\mathbb{C}^n$. We shall treat this topic briefly in Section 3, so we only say a few words about it here.

Caldéron and Zygmund establish the existence of nontangential boundary limits for holomorphic functions with a growth estimate on certain domains (including the unit ball) in $\mathbb{C}^n$. These are the same boundary limit theorems that one can prove for harmonic functions in that context. It is now understood that holomorphic functions of several complex variables are special. The real part of a holomorphic function in that context is harmonic, to be sure. But it is in fact *pluriharmonic*, and that is a much more special property. It turns out that holomorphic functions of several complex variables have boundary limits through much broader approach regions than nontangential (see [15] for a thorough and authoritative treatment of this phenomenon). These regions are called *admissible approach regions*, and have been the subject of intense study. We shall treat them in detail in Section 6 below.

3. The modern era

The groundbreaking result, from our point of view, in the harmonic analysis of several complex variables was the work in [38,39] by Koranyi. In these papers, Koranyi studies the boundary behavior of Hardy space functions f in H^2 of the unit ball B. Write

$$B = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1\}.$$

Set, for $0 < p < \infty$,

$$\begin{aligned} H^2(B) = \Bigg\{ & f \text{ holomorphic on } B : \sup_{0<r<1} \left(\int_{\partial B} |f(r\zeta)|^p d\sigma(\zeta) \right)^{1/p} \\ & \equiv \|f\|_{H^p(B)} < \infty \Bigg\}, \end{aligned}$$

where $d\sigma$ is rotationally invariant area measure on ∂B. Let $H^\infty(B)$ be the space of bounded holomorphic functions on the ball, equipped with the obvious norm.

Let $P \in \partial B, \alpha > 1$, and set

$$\mathcal{A}_\alpha(P) = \{z \in B : |1 - z \cdot \overline{P}| < \alpha(1 - |z|)\}.$$

This "approach region" at $P \in \partial B$ should be compared and contrasted with the more classical

$$\Gamma_\alpha(P) = \{z \in B : |z - P| < \alpha(1 - |z|)\}.$$

The approach region $\mathcal{A}_\alpha(P)$ allows for parabolic approach in certain directions. Both these approach regions "touch" the boundary at P. However, they give different geometric means for approaching P.

Theorem 3.1. *Let $f \in H^2(B)$. Then, for σ-almost every $\zeta \in \partial B$, we have that the limit*

$$\widetilde{f}(\zeta) \equiv \lim_{\Gamma_\alpha(\zeta) \ni z \to \zeta} f(z)$$

exists. We also have that the limit

$$\widetilde{f}(\zeta) \equiv \lim_{\mathcal{A}_\alpha(\zeta) \ni z \to \zeta} f(z)$$

exists.

The first of these results, as has been indicated above, was proved by Caldéron and Zygmund. The second is due to Adam Koranyi [38,39].

It had been a matter of commonly held belief, going back to J.E. Littlewood, Walter Rudin, and others, that nontangential approach was the optimal approach for a Fatou-type theorem on the boundary limits of Hardy space functions. Numerous well-known examples illustrated the point (see [15] for details). So Koranyi's result came as something of a shock. How did he establish this remarkable theorem?

The methodology is fundamental and important. As illustrated in Chapter 8 of [40], the key fact used to prove the classical result (on the disc) about nontangential boundary behavior is that the Poisson integral is bounded above by the Hardy–Littlewood maximal function:

Theorem 3.2. *If $\zeta \in \partial D, 1 < \alpha < \infty$, then there is a constant $C_\alpha > 0$ such that, if $f \in L^1(\partial D)$, then*

$$\sup_{re^{i\phi} \in \Gamma_\alpha(\zeta)} |P_r f(e^{i\phi})| \le C_\alpha M f(\zeta).$$

Proof. For $re^{i\phi} \in \Gamma_\alpha(P)$, we have

$$|\theta - \phi| \le 2\alpha(1-r).$$

Therefore, for $1/\alpha \le r < 1$, we obtain

$$\begin{aligned}
|P_r f(e^{i\phi})| &= \left| \frac{1}{2\pi} \int_0^{2\pi} f(e^{i(\phi-\psi)}) \frac{1-r^2}{1-2r\cos\psi + r^2} d\psi \right| \\
&= \left| \frac{1}{2\pi} \int_0^{2\pi} f(e^{i(\phi-\psi)}) \frac{1-r^2}{(1-r)^2 + 2r(1-\cos\psi)} d\psi \right| \\
&\le \frac{4}{2\pi} \sum_{j=0}^{\log_2(\pi/\alpha(1-r))} \int_{S_j} |f(e^{i(\phi-\psi)})| \frac{1-r^2}{(1-r)^2 + 2r(2^{j-1}\alpha(1-r))^2} d\psi \\
&\quad + \frac{1}{2\pi} \int_{|\psi|<\alpha(1-r)} |f(e^{i(\phi-\psi)})| \frac{1-r^2}{(1-r)^2} d\psi,
\end{aligned}$$

where $S_j = \{\psi : 2^j\alpha(1-r) \le |\psi| < 2^{j+1}\alpha(1-r)\}$. Now this is

$$\begin{aligned}
&\le \frac{4\alpha}{4\pi\alpha^2} \sum_{j=0}^{\infty} \frac{1}{2^{2j-2}(1-r)} \int_{|\psi|<(2+2^{j+1})\alpha(1-r)} |f(e^{i(\theta-\psi)})| d\psi \\
&\quad + \frac{2}{2\pi} \frac{1}{1-r} \int_{|\psi|<3\alpha(1-r)} |f(e^{i(\theta-\psi)})| d\psi
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{32}{\pi}\sum_{j=0}^{\infty} 2^{-j}\left[\frac{1}{2\alpha(2+2^{j+1})(1-r)}\int_{|\psi|<(2+2^{j+1})\alpha(1-r)} |f(e^{i(\theta-\psi)})|d\psi\right] \\
&\quad + \frac{6\alpha}{\pi}\frac{1}{2\cdot 3\alpha(1-r)}\int_{|\psi|<3\alpha(1-r)} |f(e^{i(\theta-\psi)})|d\psi \\
&\leq \frac{32}{\pi}\cdot\sum_{j=0}^{\infty} 2^{-j} Mf(\theta) + \frac{6\alpha}{\pi} Mf(\theta) \\
&\leq \frac{64}{\pi} Mf(\theta) + \frac{6\alpha}{\pi} Mf(\theta).
\end{aligned}$$

If $0 < r \leq 1/\alpha$, then

$$\begin{aligned}
|P_r f(\phi)| &\leq \frac{1}{2\pi}\int_0^{2\pi} |f(e^{i(\phi-\psi)})|(2\alpha/(\alpha-1))d\psi \\
&\leq \frac{2\alpha}{\alpha-1} Mf(\theta). \quad \square
\end{aligned}$$

Here M is the Hardy–Littlewood maximal function (defined below) and $P_r f$ is the Poisson integral. Koranyi's strategy was to take advantage of the multivariable complex-analytic setting by using a different kernel. He could have used the Poisson kernel for the unit ball in $\mathbb{R}^{2n} \approx \mathbb{C}^n$, but that would have only given him nontangential convergence – nothing new. He instead used the Poisson–Szegő kernel, which has a different sort of singularity than the Poisson kernel. And the Poisson–Szegő integral is in turn dominated by a *different* maximal function. Let us now consider the details.

As explained in Section 1.5 of [40], the Szegő kernel for the unit ball in $\mathbb{C}^n$ is given by

$$S(z,\zeta) = \frac{(n-1)!}{2\pi^n}\,\frac{1}{(1-z\cdot\overline{\zeta})^n}.$$

Here c_n is a constant that depends only on the complex dimension of the ambient space. This kernel is the canonical reproducing kernel for the Hardy space $H^2(B)$. See also our Section 9.

Now a classical construction of Hua [32] gives rise to a new reproducing kernel which is positive. Namely, we set

$$\mathcal{P}(z,\zeta) = \frac{|S(z,\zeta)|^2}{S(z,z)}.$$

It is a straightforward calculation to see that integration against $\mathcal{P}(z,\zeta)$ reproduces elements of $C(\overline{B}) \cap \mathcal{O}(B)$ (that, is functions holomorphic on B that extend to be continuous on $\overline{B}$) – see [42] for more on these matters. The explicit formula for $\mathcal{P}$ on the ball is

$$\mathcal{P}(z,\zeta) = \frac{(n-1)!}{2\pi^n}\,\frac{(1-|z|^2)^n}{|1-z\cdot\overline{\zeta}|^{2n}}.$$

We see that the singularity of the Poisson–Szegő kernel is the nonisotropic expression that we encountered in the definition of $\mathcal{A}_\alpha(P)$. In short, the classical Hardy–Littlewood maximal operator based on round balls fits the classical Poisson kernel because the singularity of that kernel is isotropic – in other words, it is *round*. By contrast, a new Hardy–Littlewood type maximal operator (which we shall define in a moment) based on nonisotropic balls will fit the Poisson–Szegő kernel.

In fact let us compare the two maximal functions. Let $P \in \partial\Omega$. If $r > 0$ then let

$$\beta_1(P, r) = \{\zeta \in \partial B : |\zeta - P| < r\}.$$

This is of course a standard, isotropic Euclidean ball intersected with ∂B. Also let

$$\beta_2(P, r) = \{\zeta \in \partial B : |1 - \zeta \cdot \overline{P}| < r\}.$$

We can plainly see that the balls β_2 are modeled on the nonisotropic geometry that we have seen before in the definition of $\mathcal{A}_\alpha(P)$. We can define a maximal function

$$M_1 f(P) = \sup_{r>0} \frac{1}{\sigma(\beta_1(P, r))} \int_{\beta_1(P,r)} |f(\zeta)| \, d\sigma(\zeta).$$

Here $d\sigma$ is $2n - 1$-dimensional Hausdorff measure on ∂B. Alternatively, we can say that $d\sigma$ is rotationally invariant area measure.

Likewise, we define

$$M_2 f(P) = \sup_{r>0} \frac{1}{\sigma(\beta_2(P, r))} \int_{\beta_2(P,r)} |f(\zeta)| \, d\sigma(\zeta).$$

The difference between the two maximal functions is the balls that are used.

Now the main point is that ∂B, equipped with area measure $d\sigma$ and the balls $\beta_1(P, r)$, is a space of homogeneous type in the sense of Coifman and Weiss [9] (see Section 3.1). And also ∂B, equipped with area measure $d\sigma$ and the balls $\beta_2(P, r)$, is a space of homogeneous type. On a space of homogeneous type, it is automatic that the corresponding maximal function, as defined above, is of weak type $(1, 1)$ and strong type (p, p) for $1 < p \le \infty$. It is a matter of direct estimation (as in Theorem 3.2 above and Section 8.1 of [40]) to see that the Poisson integral is majorized by M_1 and the Poisson–Szegő integral is majorized by M_2. The rest of the Fatou theorem is standard and well-known machinery.

3.1. Spaces of homogeneous type

These are fundamental ideas of K.T. Smith [68] and L. Hörmander [31] which were later developed by R.R. Coifman and Guido Weiss [9] into a coherent theory.

Definition 3.3. We call a set X a *space of homogeneous type* if it is equipped with a collection of open balls $B(x, r)$ and a Borel regular measure μ, together with positive constants C_1, C_2, such that

(3.3.1) The Positivity Property: $0 < \mu(B(x, r)) < \infty$ for $x \in X$ and $r > 0$;
(3.3.2) The Doubling Property: $\mu(B(x, 2r)) \le C_1 \mu(B(x, r))$ for $x \in X$ and $r > 0$;
(3.3.3) The Enveloping Property: If $B(x, r) \cap B(y, s) \ne \emptyset$ and $r \ge s$, then $B(x, C_2 r) \supseteq B(y, s)$.

We frequently use the notation (X, μ) to denote a space of homogeneous type. In some contexts a space of homogeneous type is equipped with a metric as well (and the balls are defined in terms of the metric), but we opt for greater generality here.

We can now define the Hardy–Littlewood maximal function on $L^1(X, \mu)$. If $f \in L^1(X, \mu)$, then define

$$Mf(x) \equiv \sup_{R>0} \frac{1}{\mu(B(x, R))} \int_{B(x,r)} |f(t)| d\mu(t).$$

Now it is a standard result from [9] that:

Proposition 3.4. *M is weak-type* (1, 1).

Since M is obviously strong type (∞, ∞) and weak type $(1, 1)$, we may apply the Marcinkiewicz interpolation theorem (see [2] or [75]) to see that M is strong type (p, p), $1 < p \leq \infty$.

4. The Fatou theory for $p < 1$

The previous section gave a way to prove Fatou-type theorems on the unit ball $B \subseteq \mathbb{C}^n$ for the space H^p when $1 \leq p \leq \infty$. For that is the range of p for which the maximal function satisfies suitable estimates. It has been a matter of some interest, from the very inception of the field, to find ways to extend the results to H^p for $0 < p < 1$.

The classical approach to the matter is by way of Blaschke products. A thorough exposition appears in [40, Section 8.1].

Matters are different in the context of several complex variables. First of all, there are no Blaschke products in several complex variables (see [66]). Thus some other tools will be needed to pass to H^p for $0 < p < 1$. Stein's idea (see [69]) is to use harmonic majorization. Details of these arguments may be founds in [40, Ch. 8].

For simplicity we continue to restrict attention to the domain the unit ball in $\mathbb{C}^n$. If $f \in H^p(B)$, then $|f|^{p/2}$ is subharmonic with a growth condition and hence certainly has a harmonic majorant h (this idea is related to Lumer's theory of Hardy spaces – see [65]). Indeed this property of harmonic majorization characterizes the Hardy spaces on an arbitrary domain in $\mathbb{C}^n$ with C^2 boundary – again see [40] as well as the next section.

Now in fact it can be shown that the harmonic majorant h satisfies the growth condition

$$\sup_{0<r<1} \left(\int_{\partial B} |h(r\zeta)|^2 d\sigma(\zeta) \right)^{1/2} \equiv \|f\|_{h^2(B)} < \infty.$$

Thus it can be shown that h has radial, indeed nontangential, boundary limits at σ-almost every boundary point of B. It follows then that h is nontangentially bounded at almost every boundary point of the ball (that the notions of nontangential limit and nontangential boundedness are equivalent is a result of Caldéron – see [5]). Thus certainly f itself is nontangentially bounded at almost every boundary point. From this it can be shown, for example, that the original holomorphic function h has nontangential boundary limits almost everywhere – again this uses ideas of Caldéron.

It remains to pass from nontangential limits to admissible limits for $0 < p \leq 1$, and we do so below in the context of more general domains Ω.

5. More general domains

We have, for convenience and simplicity, concentrated in the preceding discussion on the domain the unit ball. The ball is convex and has a great deal of symmetry, and that makes several of the arguments much simpler. In particular, we can take advantage of dilations and also explicit formulas for certain kernels. But, especially in several complex variables, there is great interest in proving results on more general domains. E.M. Stein in [69] laid the foundations for the study of such domains. In particular, he noted that the classical idea of taking a Hardy space function f on the disc or ball and associating to it a family $f_r(z) = f(rz), 0 < r < 1$, can be replaced by something more geometric. The idea is this.

Let $\Omega \subseteq \mathbb{R}^N$ be a smoothly bounded domain. We cover Ω by finitely many domains $\Omega_1, \ldots, \Omega_k$ with the following properties:

(4.1) $\Omega = \cup_j \Omega_j$;
(4.2) For each j, the set $\partial\Omega \cap \partial\Omega_j$ is an $(N-1)$-dimensional manifold with boundary;
(4.3) There is an $\epsilon_0 > 0$ and a vector ν_j transversal to $\partial\Omega \cap \partial\Omega_j$ and pointing out of Ω such that $\Omega_j - \epsilon\nu_j \equiv \{z - \epsilon\nu_j : z \in \Omega_j\} \subset\subset \Omega$, all $0 < \epsilon < \epsilon_0$.

The proof that the Ω_j exist is an exercise in elementary geometric analysis.

We close this section with a very basic result about the boundary behavior of holomorphic functions on fairly general domains in $\mathbb{C}^n$. For the proof, see [40, p. 347]. A clever argument of Lempert [53] gives the next key result:

Proposition 5.1. *Let $\Omega \subset\subset \mathbb{C}^n$ have C^2 boundary. Let $0 < p < \infty$ and $f \in H^p(\Omega)$. Write $\Omega = \cup_{j=1}^k \Omega_j$ as in* (4.1) *through* (4.3) *, and let $\nu_1, \ldots, \nu_k$ be the associated normal vectors. Then, for each $j \in \{1, \ldots, k\}$, it holds that*

$$\lim_{\epsilon\to 0^+} f(\zeta - \epsilon\nu_j) \equiv \widetilde{f}(\zeta)$$

exists for σ-almost every $\zeta \in \partial\Omega_j \cap \partial\Omega$.

Explorations of the sharp forms of Fatou theorems on pseudoconvex domains appear in [56].

6. The generalization to admissible convergence

On the ball we can get decisive information for H^p, $1 < p < \infty$ by exploiting the Poisson–Szegő kernel. Unfortunately, there is little information about the Poisson–Szegő kernel on general domains in $\mathbb{C}^n$. Using incisive ideas of Fefferman [16], one *can* obtain an asymptotic expansion for the Poisson–Szegő kernel on a strongly pseudoconvex domain in $\mathbb{C}^n$. Then one could imitate Koranyi's proof to obtain admissible boundary limits for H^p functions on a strongly pseudoconvex domain. Nobody has ever carried out the details of this program, but it is feasible. For more general domains, even finite type domains in $\mathbb{C}^2$, matters are much less clear.

Stein, however, in his seminal work [69], produced a completely different and quite original approach to the matter. He proved a result on all bounded domains with C^2 boundary. His result is sharp only on strongly pseudoconvex domains. But it contributes important

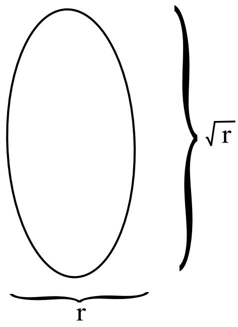

Fig. 1. A nonisotropic ball.

information on any domain. S. Ross Barker was able [1] to simplify Stein's arguments considerably.

Of course an isotropic ball $\beta_1(p, r)$ has σ-measure approximately r^{2n-1}. But our earlier calculations show that the nonisotropic ball $\beta_2(P, r)$ has σ-measure approximately r^n (see the details below). This will make the analysis with the β_2 decidedly different. In short, the balls we are considering have dimension $\sim r$ in the complex space containing the normal vector ν and dimension $\sim\sqrt{r}$ in the orthogonal complement (see Fig. 1). [Refer to the more detailed discussion below.] The word "non-isotropic" means that we have different geometric behavior in different directions.

In the classical setup, we considered *cones* modeled on the balls β_1:

$$\Gamma_\alpha(P) = \{z \in B : |z - P| < \alpha(1 - |z|)\}, \quad P \in \partial B,\ \alpha > 1.$$

In the new situation we consider *admissible regions* modeled on the balls β_2:

$$\mathcal{A}_\alpha(P) = \{z \in B : |1 - z \cdot \overline{P}| < \alpha(1 - |z|)\}.$$

Our new theorem about boundary limits of H^p functions is as follows:

Theorem 6.1. *Let $f \in H^p(B)$, $0 < p \leq \infty$. Let $\alpha > 1$. Then the limit*

$$\lim_{\mathcal{A}_\alpha(P)\ni z\to P} f(P) \equiv \widetilde{f}(P)$$

exists for σ-almost every $P \in \partial B$.

Since the Poisson–Szegö kernel is known explicitly on the ball, then for $p \geq 1$ the proof is deceptively straightforward: Let M_2 be the nonisotropic maximal function as before. Also set $f(z) = \int_{\partial B} \mathcal{P}(z, \zeta) f(\zeta)\, d\sigma(\zeta)$ for $z \in B$. Then, by explicit computation similar to the proof of Theorem 3.2,

$$f_2^{*,\alpha}(P) \equiv \sup_{z\in\mathcal{A}_\alpha(P)} |f(z)| \leq C_\alpha M_2 f(P), \quad \text{all } f \in L^1(\partial B).$$

This crucial fact, together with appropriate estimates on the operator M_2, enables one to complete the proof along classical lines for $p \geq 1$. For $p < 1$, matters are more subtle.

Now we pass to more general domains. Let $\Omega \subseteq \mathbb{C}^n$ be smoothly bounded. If z, w are vectors in $\mathbb{C}^n$, we continue to write $z \cdot \overline{w}$ to denote $\sum_j z\overline{w}_j$. Also, for $\Omega \subseteq \mathbb{C}^n$ a domain with C^2 boundary, $P \in \partial\Omega$, we let ν_P be the unit outward normal vector at P. Let $\mathbb{C}\nu_P$ denote the complex line generated by $\nu_P : \mathbb{C}\nu_P = \{\zeta\nu_P : \zeta \in \mathbb{C}\}$.

By dimension considerations, if $T_P(\partial\Omega)$ is the $(2n-1)$-dimensional real tangent space to $\partial\Omega$ at P, then $\ell = \mathbb{C}\nu_P \cap T_P(\partial\Omega)$ is a (one-dimensional) real line. Let

$$\begin{aligned}\mathcal{T}_P(\partial\Omega) &= \{z \in \mathbb{C}^n : z \cdot \overline{\nu}_P = 0\} \\ &= \{z \in \mathbb{C}^n : z \cdot \overline{w} = 0 \, \forall \, w \in \mathbb{C}\nu_P\}.\end{aligned}$$

A fortiori, $\mathcal{T}_P(\partial\Omega) \subseteq T_P(\partial\Omega)$. If $z \in \mathcal{T}_P(\partial\Omega)$, then $iz \in \mathcal{T}_P(\partial\Omega)$. Therefore $\mathcal{T}_P(\partial\Omega)$ may be thought of as an $(n-1)$-dimensional complex subspace of $T_P(\partial\Omega)$. Clearly, $\mathcal{T}_P(\partial\Omega)$ is the complex subspace of $T_P(\partial\Omega)$ of maximal dimension. It contains all complex subspaces of $T_P(\partial\Omega)$. We may think of $\mathcal{T}_P(\partial\Omega)$ as the real orthogonal complement in $T_P(\partial\Omega)$ of ℓ.

The next definition is best understood in light of the foregoing discussion and the definition of $\beta_2(P, r)$ in the boundary of the unit ball B. Let $\Omega \subset\subset \mathbb{C}^n$ have C^2 boundary. For $P \in \partial\Omega$, let $\pi_P : \mathbb{C}^n \to \mathcal{N}_P$ be (real or complex) orthogonal projection. Here $\mathcal{N}_P$ is the space that is Hermitian orthogonal to $\mathcal{T}_P$. $\square$

Definition 6.2. If $P \in \partial\Omega$ let

$$\begin{aligned}\beta_1(P, r) &= \{\zeta \in \partial\Omega : |\zeta - P| < r\}; \\ \beta_2(P, r) &= \{\zeta \in \partial\Omega : |\pi_P(\zeta - P)| < r, |\zeta - P| < r^{1/2}\}.\end{aligned}$$

The ball $\beta_2(P, r)$ has diameter $\sim\sqrt{r}$ in the $(2n-2)$ complex tangential directions and diameter $\sim r$ in the one (normal) direction. Therefore $\sigma(\beta_2(P, r)) \approx (\sqrt{r})^{2n-2} \cdot r \approx Cr^n$.

If $z \in \Omega$, $P \in \partial\Omega$, we let

$$\delta_P(z) = \min\{\mathrm{dist}(z, \partial\Omega), \mathrm{dist}(z, T_P(\Omega))\}.$$

Notice that, if Ω is convex, then $\delta_P(z) = \delta_\Omega(z)$.

Definition 6.3. If $P \in \partial\Omega, \alpha > 1$, let

$$\mathcal{A}_\alpha = \{z \in \Omega : |(z - P) \cdot \overline{\nu}_P| < \alpha\delta_P(z), |z - P|^2 < \alpha\delta_P(z)\}.$$

Notice that δ_P is used because near non-convex boundary points we still want $\mathcal{A}_\alpha$ to have the fundamental geometric shape of (paraboloid $\times$ cone) as shown in Fig. 2.

Definition 6.4. If $f \in L^1(\partial\Omega)$ and $P \in \partial\Omega$ then we define

$$M_j f(P) = \sup_{r>0} \sigma(\beta_j(P, r))^{-1} \int_{\beta_j(P,r)} |f(\zeta)| d\sigma(\zeta), \quad j = 1, 2.$$

Definition 6.5. If $f \in C(\Omega)$, $P \in \partial\Omega$, then we define

$$f_2^{*,\alpha}(P) = \sup_{z \in \mathcal{A}_\alpha(P)} |f(z)|.$$

The first step of our program is to prove an estimate for M_2. This will require a covering lemma (indeed, it is known that weak type estimates for operators like M_j are logically

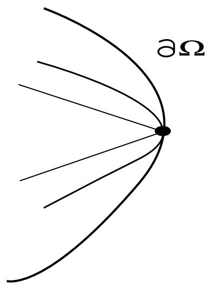

Fig. 2. The admissible approach region.

equivalent to covering lemmas – see A. Cordoba and R. Fefferman [10]). We exploit the structure of a domain of homogeneous type, which is explained in Section 3.1.

Thus we need to see that the $\beta_2(P, r)$ on $X = \partial\Omega$ with $m = \sigma$ satisfy axioms of a space of homogeneous type. The first axiom is trivial. The second is easy if one uses the fact that $\partial\Omega$ is C^2 and compact (use the Remark following Theorem 6.1). Thus it remains to check the third axiom (in many applications, this is the most difficult property to check). We refer the reader to [40, Chapter 8] for the details.

Now, from the general theory of spaces of homogeneous type, we have

Corollary 6.6. *If $f \in L^1(\partial\Omega)$, then*

$$\sigma\{\zeta \in \omega : M_2 f(\zeta) > \lambda\} \le C\frac{\|f\|_{L^1(\partial\Omega)}}{\lambda}, \quad \textit{all } \lambda > 0.$$

Corollary 6.7. *The operator M_2 maps $L^2(\partial\Omega)$ to $L^2(\partial\Omega)$ boundedly.*

The next lemma is the heart of the matter: it is the technical device that allows us to estimate the behavior of a holomorphic function in the interior (in particular, on an admissible approach region) in terms of a maximal function on the boundary. The argument comes from Stein [69] and Barker [1].

Lemma 6.8. *Let $u \in C(\overline{\Omega})$ be non-negative and plurisubharmonic on Ω. Define $f = u|_{\partial\Omega}$. Then*

$$u_2^{*,\alpha}(P) \le C_\alpha M_2(M_1 f)(P)$$

for all $P \in \partial\Omega$ and any $\alpha > 1$.

Proof. After rotating and translating coordinates, we may suppose that $P = 0$ and $\nu_P = (1 + i0, 0, \ldots, 0)$. Let $\alpha' > \alpha$. Then there is a small positive constant k such that if $z = (x_1 + iy_1, z_2, \ldots, z_n) \in \mathcal{A}_\alpha(P)$ then $\mathcal{D}(z) = D(z_1, -kx_1) \times D^{n-1}((z_2, \ldots, z_n), \sqrt{-kx_1}) \subseteq \mathcal{A}_{\alpha'}(P)$ (see Fig. 3).

We restrict attention to $z \in \Omega$ so close to $P = 0$ that the projection along ν_P given by

$$z = (x_1 + iy_1, \ldots, x_n + iy_n) \to (\tilde{x}_1 + i\tilde{y}_1, x_2 + iy_2, \ldots, x_n + iy_n) \equiv \tilde{z} \in \partial\Omega$$

makes sense. [Observe that points z that are far from $P = 0$ are trivial to control using our estimates on the Poisson kernel.]

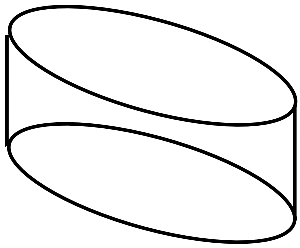

Fig. 3. The polydisc that reflects the shape of the domain.

The projection of $\mathcal{D}(z)$ along ν_P into the boundary lies in a ball of the form $\beta_2(\tilde{z}, Kx_1)$ – *this observation is crucial.*

Notice that the subharmonicity of u implies that $u(z) \leq Pf(z)$. Also there is a $\beta > 1$ such that $z \in \mathcal{A}_{\alpha'}(0) \Rightarrow z \in \mathbf{1},_\beta(\tilde{z})$. Therefore the standard argument in Theorem 3.2 yields that

$$|u(z)| \leq |Pf(z)| \leq C_\alpha M_1 f(\tilde{z}). \tag{6.8.1}$$

Now we bring the complex analysis into play. For we may exploit the plurisubharmonicity of $|u|$ on $\mathcal{D}(z)$ by invoking the sub-averaging property in each dimension in succession. Thus

$$\begin{aligned} |u(z)| &\leq \left(\pi |kx_1|^2\right)^{-1} \cdot \left(\pi (\sqrt{-kx_1})^2\right)^{-(n-1)} \int_{\mathcal{D}(z)} |u(\zeta)| dV(\zeta) \\ &= Cx_1^{-n-1} \int_{\mathcal{D}(z)} |u(\zeta)| dV(\zeta). \end{aligned}$$

Notice that if $z \in \mathcal{A}_\alpha(P)$ then each ζ in the last integrand is in $\mathcal{A}_{\alpha'}(P)$. Thus the last line is

$$\begin{aligned} &\leq C' x_1^{-n-1} \int_{\mathcal{D}(z)} M_1 f(\tilde{\zeta}) dV(\zeta) \\ &\leq C'' x_1^{-n-1} \cdot x_1 \int_{\beta_2(\tilde{z}, Kx_1)} M_1 f(t) d\sigma(t) \\ &\leq C''' x_1^{-n} \int_{\beta_2(0, K'x_1)} M_1 f(t) d\sigma(t) \\ &\leq C'''' \left(\sigma\left(\beta_2(0, K'x_1)\right)\right)^{-1} \int_{\beta_2(0, K'x_1)} M_1 f(t) d\sigma(t) \\ &\leq C'''' M_2(M_1 f)(0). \quad \square \end{aligned}$$

Now we have our main result:

Theorem 6.9. *Let $0 < p \leq \infty$. Let $\alpha > 1$. If $\Omega \subset\subset \mathbb{C}^n$ has C^2 boundary and $f \in H^p(\Omega)$, then for σ-almost every $P \in \partial\Omega$ we have*

$$\lim_{\mathcal{A}_\alpha(P) \ni z \to P} f(z)$$

exists.

It is natural to ask whether C^2 boundary is really necessary for the results being discussed here. If one is only interested in nontangential convergence, then Lipschitz boundary is sufficient. For with Lipschitz boundary it is straightforward to check that $\partial\Omega$ equipped with the standard, isotropic Euclidean balls is still a space of homogeneous type. See Section 3.1. Thus the Hardy–Littlewood maximal function is still weak type (1, 1) and the arguments go through as before. However, if we want to show that the nonisotropic balls form a space of homogeneous type, then some boundary smoothness is needed. Even if the boundary is $C^{2-\epsilon}$, it is not clear that the enveloping property (property 3.3.3) of a space of homogeneous type is satisfied.

There are ways to define balls in the boundary as the projections of Kobayashi metric balls in the interior (see [46]). This requires less boundary smoothness, and it appears that one can obtain results (at least in the strongly pseudoconvex case) with just C^1 boundary.

7. Singular integrals

In many ways the heart of modern harmonic analysis is the theory of singular integrals. Most any question in the classical theory of Fourier series can be rendered as a problem about the Hilbert transform (which is the only singular integral in dimension one) or the maximal Hilbert transform. In higher dimensions there are many singular integrals. See [47] or [72] for background on these ideas.

It is natural to wonder what are the right singular integrals for the study of the function theory of several complex variables, and how such integrals might be characterized. A full answer to this question is not known at this time. But, at least in some contexts, there are fairly complete answers.

In fact the best understood venue for Fourier analysis and singular integrals in several complex variables is the boundary of the unit ball B in $\mathbb{C}^n$. And the reason for this is fairly straightforward. Just as the unit disc in the plane may be identified (via the Cayley transform) with the upper halfplane, and the boundary of the disc identified with the line, so the ball in $\mathbb{C}^n$ may be identified (by a generalized Cayley transform) with a "Siegel upper halfspace" and the boundary of the ball identified with its parabolic boundary. That boundary, in turn, may be mapped in a natural fashion to the Heisenberg group. Thus the boundary may be equipped with both translations (in the Heisenberg group structure) and (nonisotropic) dilations. So some Fourier analysis, and certainly a theory of singular integrals, is feasible. We shall provide some of the details of these ideas in what follows in this section and the next.

Analysis on the Heisenberg group is fascinating because the group is topologically Euclidean but analytically non-Euclidean. Many of the most basic ideas of analysis must be developed again from scratch.

7.1. The role of the Heisenberg group in complex analysis

We would like to analyze the unit ball $B = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_1|^2 + \cdots |z_n|^2 < 1\}$ in a fashion similar to what is commonly done on the unit disc in the plane (see [30]). It turns out that, in this situation, the unbounded realization (see [34]) of the domain B is given by

$$\mathcal{U} = \left\{ (w_1, \ldots, w_n) \in \mathbb{C}^n : \operatorname{Im} w_1 > \sum_{j=2}^{n} |w_j|^2 \right\}.$$

It is convenient to write $w' \equiv (w_2, \ldots, w_n)$. We refer to our domain $\mathcal{U}$ as the *Siegel upper half space*, and we write its defining equation as $\operatorname{Im} w_1 > |w'|^2$.

Now the mapping that shows B and $\mathcal{U}$ to be biholomorphically equivalent is given by

$$\begin{aligned}\Phi &: B \longrightarrow \mathcal{U}\\ (z_1, \ldots, z_n) &\longmapsto \left(i \cdot \frac{1 - z_1}{1 + z_1}, \frac{z_2}{1 + z_1}, \ldots, \frac{z_n}{1 + z_1} \right).\end{aligned}$$

It is immediate to calculate that

$$\Phi^{-1}(w) = \left(\frac{i - w_1}{i + w_1}, \frac{2iw_2}{i + w_1}, \ldots, \frac{2iw_n}{i + w_n} \right).$$

If $\Omega \subseteq \mathbb{C}^n$ is any domain, we let $\operatorname{Aut}(\Omega)$ denote the collection of biholomorphic self-mappings of Ω. This set forms a group when equipped with the binary operation of composition of mappings. In fact it is a topological group with the topology of uniform convergence on compact sets (which is the same as the compact-open topology). There is a natural isomorphism between $\operatorname{Aut}(B)$ and $\operatorname{Aut}(\mathcal{U})$ given by

$$\operatorname{Aut}(B) \ni \varphi \longmapsto \Phi \circ \varphi \circ \Phi^{-1} \in \operatorname{Aut}(\mathcal{U}). \tag{7.1.1}$$

It turns out that we can understand the automorphism group of B more completely by sometimes passing to the automorphism group of $\mathcal{U}$. We shall use the idea of the Iwasawa decomposition $G = KAN$. Here K is the compact part of G, A is the abelian part of G, and N is the nilpotent part of G.

The compact part K of $\operatorname{Aut}(B)$ is the collection of all automorphisms that fix the origin. It is easy to prove, using a version of the Schwarz lemma (see [65]), that any such automorphism is a unitary rotation.

Now let us look at the abelian piece of $\operatorname{Aut}(\mathcal{U})$. For this part, it is most convenient to begin our analysis on $\mathcal{U}$. Let us consider the group of dilations, which consists of the nonisotropic mappings

$$\widetilde{\alpha}_\delta : \mathcal{U} \longrightarrow \mathcal{U}$$

given by

$$\widetilde{\alpha}_\delta(w_1, \ldots, w_n) = (\delta^2 w_1, \delta w_2, \delta w_3, \ldots, \delta w_n) \tag{7.1.2}$$

for any $\delta > 0$. Check for yourself that $\widetilde{\alpha}_\delta$ maps $\mathcal{U}$ to $\mathcal{U}$. We call these mappings nonisotropic (meaning "acts differently in different directions") because they treat the w_1 variable differently from the $w_2, \ldots, w_n$ variables. The group is clearly abelian. It corresponds, under the mapping Φ, to the group of mappings on B given by

$$\alpha_\delta(z_1, \ldots, z_n) = \Phi^{-1} \circ \widetilde{\alpha}_\delta \circ \Phi(z).$$

Of course it is just a tedious algebra exercise to determine α_δ:

$$\begin{aligned}\alpha_\delta(z) = \Bigg(&\frac{(1 - \delta^2) + z_1(1 + \delta^2)}{(1 + \delta^2) + z_1(1 - \delta^2)}, \frac{2\delta z_2}{(1 + \delta^2) + z_1(1 - \delta^2)}, \ldots, \\ &\frac{2\delta z_n}{(1 + \delta^2) + z_1(1 - \delta^2)} \Bigg).\end{aligned}$$

7.1.1. The Heisenberg group and its action on $\mathcal{U}$

The *Heisenberg group* of order $n-1$, denoted $\mathbb{H}_{n-1}$, is an algebraic structure that we impose on $\mathbb{C}^{n-1} \times \mathbb{R}$. Let (ζ, t) and (ξ, s) be elements of $\mathbb{C}^{n-1} \times \mathbb{R}$. Then the binary Heisenberg group operation is given by

$$(\zeta, t) \cdot (\xi, s) = (\zeta + \xi, t + s + 2\mathrm{Im}\,(\zeta \cdot \overline{\xi})).$$

It is clear, because of the Hermitian inner product $\zeta \cdot \overline{\xi} = \zeta_1 \overline{\xi}_1 + \cdots + \zeta_{n-1}\overline{\xi}_{n-1}$, that this group operation is non-abelian (although in a fairly subtle fashion).

Now an element of $\partial\mathcal{U}$ has the form $(\mathrm{Re}\, w_1 + i|(w_2, \ldots, w_n)|^2, w_2, \ldots, w_n) = (\mathrm{Re}\, w_1 + i|w'|^2, w')$, where $w' = (w_2, \ldots, w_n)$. We identify this boundary element with the Heisenberg group element $(w', \mathrm{Re}\, w_1)$, and we call the corresponding mapping Ψ. Now we can specify how the Heisenberg group acts on $\partial\mathcal{U}$. If $w = (w_1, w') \in \partial\mathcal{U}$ and $g = (z', t) \in \mathbb{H}_{n-1}$ then we have the action

$$g[w] = \Psi^{-1}[g \cdot \Psi(w)] = \Psi^{-1}[g \cdot (w', \mathrm{Re}\, w_1)] = \Psi^{-1}[(z', t) \cdot (w', \mathrm{Re}\, w_1)].$$

More generally, if $w \in \mathcal{U}$ is *any element* then we write

$$\begin{aligned} w &= (w_1, w_2, \ldots, w_n) = (w_1, w') \\ &= \Big((\mathrm{Re}\, w_1 + i|w'|^2) + i(\mathrm{Im}\, w_1 - |w'|^2), w_2, \ldots, w_n\Big) \\ &= \Big(\mathrm{Re}\, w_1 + i|w'|^2, w'\Big) + \Big(i(\mathrm{Im}\, w_1 - |w'|^2), 0, \ldots, 0\Big). \end{aligned}$$

The first expression in parentheses is an element of $\partial\mathcal{U}$. It is convenient to let $\rho(w) = \mathrm{Im}\, w_1 - |w'|^2$. We think of ρ as a "height function". In short, we are expressing an arbitrary element $w \in \mathcal{U}$ as an element in the boundary plus a translation "up" to a certain height in the i direction of the first variable.

Now we let g act on w by

$$\begin{aligned} g[w] &= g\Big[\Big(\mathrm{Re}\, w_1 + i|w'|^2, w'\Big) + \Big(i\rho(w), 0, \ldots, 0\Big)\Big] \\ &\equiv g\Big[\Big(\mathrm{Re}\, w_1 + i|w'|^2, w'\Big)\Big] + \Big(i\rho(w), 0, \ldots, 0\Big). \end{aligned} \tag{7.1.1.1}$$

In other words, we let g act on level sets of the height function.

One can calculate that

$$g[w] = (t + i|z'|^2 + w_1 + i2\overline{z}' \cdot w', z' + w').$$

This mapping is plainly holomorphic in w (but *not* in z!).

As we have mentioned previously, the Heisenberg group acts simply transitively on the boundary of $\mathcal{U}$. Thus the group may be identified with the boundary in a natural way. Let us now make this identification explicit. First observe that $\mathbf{0} \equiv (0, \ldots, 0) \in \partial\mathcal{U}$. If $g = (z', t) \in \mathbb{H}_{n-1}$ then

$$g[\mathbf{0}] = \Psi^{-1}[(z', t) \cdot (0', 0)] = \Psi^{-1}[(z', t)] = (t + i|z'|^2, z') \in \partial\mathcal{U}.$$

Conversely, if $(\mathrm{Re}\, w_1 + i|w'|^2, w') \in \partial\mathcal{U}$ then let $g = (w', \mathrm{Re}\, w_1)$. Hence

$$g[\mathbf{0}] = \Psi^{-1}[(w', \mathrm{Re}\, w_1)] = (\mathrm{Re}\, w_1 + i|w'|^2, w') \in \partial\mathcal{U}.$$

The upshot of the calculations in this subsubsection is that analysis on the boundary of the ball B may be reduced to analysis on the boundary of the Siegel upper half space $\mathcal{U}$. And that in turn is equivalent to analysis on the Heisenberg group $\mathbb{H}_{n-1}$. The Heisenberg group is a step-one nilpotent Lie group.

7.1.2. Distinguished 1-parameter subgroups of the Heisenberg group

The Heisenberg group $\mathbb{H}^{n-1}$ has $2n-1$ real dimensions and we can define the differentiation of a function in each direction consistent with the group structure by considering 1-parameter subgroups in each direction.

Let $g = [z, t] \in \mathbb{H}^{n-1}$, where $z = (z_1, \ldots, z_{n-1}) = (x_1 + iy_1, \ldots, x_{n-1} + iy_{n-1})$ and $t \in \mathbb{R}$. If we let

$$\begin{aligned}\gamma_{2j-1}(s) &= [(0, \ldots, s + i0, \ldots, 0), 0]\\ \gamma_{2j}(s) &= [(0, \ldots, 0 + is, \ldots, 0), 0]\end{aligned}$$

for $1 \le j \le n-1$ and the s term in the jth slot, and if we let

$$\gamma_{2n-1}(s) = \gamma_t(s) = [0, s]$$

[with $(n-1)$ zeros and one s], then each forms a one-parameter subgroup of $\mathbb{H}^n$.

We define the differentiation of f at $g = [z, t]$ in each one-parameter group direction as follows:

$$\begin{aligned}X_j f(g) &\equiv \frac{d}{ds} f(g \cdot \gamma_{2j-1}(s))\bigg|_{s=0} = \left(\frac{\partial f}{\partial x_j} + 2y_j \frac{\partial f}{\partial t}\right)[z, t], \quad 1 \le j \le n-1,\\ Y_j f(g) &\equiv \frac{d}{ds} f(g \cdot \gamma_{2j}(s))\bigg|_{s=0} = \left(\frac{\partial f}{\partial y_j} - 2x_j \frac{\partial f}{\partial t}\right)[z, t], \quad 1 \le j \le n-1,\\ T f(g) &\equiv \frac{d}{ds} f(g \cdot \gamma_t(s))\bigg|_{s=0} = \frac{\partial f}{\partial t}[z, t].\end{aligned}$$

We think of X_j, Y_j, and T as vector fields on the Heisenberg group.

7.1.3. Commutators in the Heisenberg group

Note that $[X_j, X_k] = [Y_j, Y_k] = [X_j, T] = [Y_j, T] = 0$ for all $1 \le j, k \le n$ and $[X_j, Y_k] = 0$ if $j \ne k$. The only nonzero commutator in the Heisenberg group is $[X_j, Y_j]$, and we calculate that right now:

$$\begin{aligned}[X_j, Y_j] &= \left(\frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}\right)\\ &\quad - \left(\frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}\right)\\ &= -4\frac{\partial}{\partial t}\\ &= -4T.\end{aligned}$$

To summarize: all commutators $[X_j, X_k]$ for $j \neq k$ and $[X_j, T]$ equal 0. The only nonzero commutator is $[X_j, Y_j] = -4T$. One upshot of these simple facts is that any *second-order commutator* $[[A, B], C]$ will be zero – just because $[A, B]$ will be either 0 or $-4T$. Thus the vector fields on the Heisenberg group form a nilpotent Lie algebra of step one.

7.1.4. Additional information about the Heisenberg group action

In $\mathbb{H}^{n-1}$, Haar measure coincides with the Lebesgue measure. [This is an easy calculation using elementary changes of variable.]

Let $g = [z, t] \in \mathbb{H}^{n-1}$. The dilation on $\mathbb{H}^{n-1}$ is defined to be

$$\alpha_\delta g = [\delta z, \delta^2 t].$$

We can easily check that α_δ is a group homomorphism.

A ball with center $[z, t]$ and radius r is defined as

$$B([z, t], r) = \{[\zeta, s] : |\zeta - z|^4 + |s - t|^2 < r^2\}.$$

For $f, g \in L^1(\mathbb{H}^n)$, we can define the convolution of f and g:

$$f * g(x) = \int f(y^{-1} \cdot x) g(y)\, dy.$$

7.2. A fresh look at classical analysis

7.2.1. The Folland–Stein theorem

Let (X, μ) be a measure space and $f : X \to \mathbb{C}$ a measurable function. We say that f is *weak-type* r, $0 < r < \infty$ if there exists some constant C such that

$$\mu\{x : |f(x)| > \lambda\} \leq \frac{C}{\lambda^r}, \qquad \text{for any } \lambda > 0.$$

Theorem 7.1 (*Folland, Stein [19]*)**.** *Let* (X, μ), (Y, ν) *be measurable spaces. Let*

$$k : X \times Y \to \mathbb{C}$$

satisfy

$$\mu\{x : |k(x, y)| > \lambda\} \leq \frac{C}{\lambda^r}, \quad (\text{for fixed } y)$$
$$\nu\{y : |k(x, y)| > \lambda\} \leq \frac{C'}{\lambda^r}, \quad (\text{for fixed } x)$$

where C *and* C' *are independent of* y *and* x *respectively and* $r > 1$. *Then*

$$f \longmapsto \int_Y f(y) k(x, y) d\nu(y)$$

maps $L^p(X)$ *to* $L^q(X)$ *where* $\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1$, *for* $1 < p < \frac{r}{r-1}$.

7.2.2. Classical Caldéron–Zygmund theory

We now wish to turn our attention to singular integral operators. One of the key tools for the classical approach to this subject is the Whitney decomposition of an open set. That is an important tool in geometric analysis that first arose in the context of the Whitney extension theorem [77]. We begin with a review of that idea.

Theorem 7.2 (*Caldéron–Zygmund decomposition*)**.** *Let f be a nonnegative, integrable function in $\mathbb{R}^N$. Then, for $\alpha > 0$ fixed, there is a decomposition of $\mathbb{R}^N$ such that*

1. $\mathbb{R}^N = F \cup \Omega$, $F \cap \Omega = \emptyset$, *$F$ is closed.*
2. $f(x) \leq \alpha$ *for almost every* $x \in F$.
3. $\Omega = \cup_j Q_j$, *where Q_j's are closed cubes with disjoint interiors and f satisfies*

$$\alpha < \frac{1}{m(Q_j)} \int_{Q_j} f(x)\, dx \leq 2^N \alpha.$$

($m(Q_j)$ denotes the measure of the cube Q_j.)

Definition 7.3 (*Caldéron–Zygmund Kernel*)**.** A Caldéron–Zygmund kernel $K(x)$ in $\mathbb{R}^N$ is one having the form

$$K = \frac{\Omega(x)}{|x|^N}$$

where

1. $\Omega(x)$ is homogeneous of degree 0
2. $\Omega(x) \in C^1(\mathbb{R}^N \setminus \{0\})$
3. $\int_\Sigma \Omega(x) d\sigma(x) = 0$, ($\Sigma$ is the unit sphere in $\mathbb{R}^N$).

The next result is the key to our study of singular integral kernels and operators.

Theorem 7.4 (*Caldéron–Zygmund*)**.** *Let $K \in L^2(\mathbb{R}^N)$. Assume that*

1. $|\widehat{K}| \leq B$
2. $K \in C^1(\mathbb{R}^N \setminus \{0\})$ *and* $|\nabla K(x)| \leq C|x|^{-N-1}$.

For $1 < p < \infty$, and $f \in L^1 \cap L^p(\mathbb{R}^N)$, set

$$Tf(x) = K * f(x) = \int_{\mathbb{R}^N} K(x-t) f(t)\, dt.$$

Then there exists a constant A_p such that

$$\|Tf\|_p \leq A_p \|f\|_p.$$

Theorem 7.5. *Let $K(x)$ be a Caldéron–Zygmund kernel. Then $f \longmapsto K * f$ is bounded on L^p, $1 < p < \infty$.*

The Caldéron–Zygmund theorem is proved using a delicate geometric analysis based on the Whitney decomposition.

7.3. Analysis on $\mathbb{H}^n$

When we were thinking of the Heisenberg group as the boundary of a domain in $\mathbb{C}^n$, then the appropriate Heisenberg group to consider was $\mathbb{H}^{n-1}$, as that Lie group has dimension $2n-1$ (the correct dimension for a boundary). Now we are about to study the Heisenberg intrinsically, in its own right, so it is appropriate (and it simplifies the notation a bit) to focus our attention on $\mathbb{H}^n$.

In $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$, the group operation is defined as

$$(z,t)\cdot(z',t') = (z+z', t+t'+2\mathrm{Im}\, z\cdot\overline{z'}), \quad z, z' \in \mathbb{C}^n,\ t, t' \in \mathbb{R}.$$

Let $g = (z,t) = (z_1, z_2, \dots, z_n, t) = (x_1+iy_1, x_2+iy_2, \dots, x_n+iy_n, t) \in \mathbb{H}^n$. We write

$$dV(g) = dx_1 dy_1 \cdots dx_n dy_n dt,$$

so that

$$dV(\delta g) = d(\delta x_1)\, d(\delta y_1) \cdots d(\delta x_n) d(\delta y_n) d\delta^2 t = \delta^{2n+2}\, dV(g).$$

We call $2n+2$ the homogeneous dimension of $\mathbb{H}^n$. [Note that the topological dimension of $\mathbb{H}^n$ is $2n+1 \neq 2n+2$.] The critical index N for a singular integral is such that

$$\int_{B(0,1)} \frac{1}{|z|^\alpha}\, dV(z) = \begin{cases} \infty & \text{if } \alpha \geq N \\ < \infty & \text{if } 0 < \alpha < N \end{cases}$$

and the critical index coincides with the homogeneous dimension. Thus the critical index for a singular integral in $\mathbb{H}^n$ is $2n+2$, which is different from the topological dimension.

7.3.1. Distance in $\mathbb{H}^n$

For $x, y \in \mathbb{H}^n$, we define the distance $d(x,y)$ as follows:

$$d(x,y) \equiv |x^{-1}\cdot y|_h.$$

Then $d(x,y)$ satisfies the following properties:

1. $d(x,y) = 0 \iff x = y$;
2. $d(x,y) = d(y,x)$;
3. $\exists \gamma_0 > 0$ such that $d(x,y) \leq \gamma_0[d(x,w) + d(w,y)]$.

7.3.2. $\mathbb{H}^n$ is a space of homogeneous type

Refer to Section 3.1. Define balls in $\mathbb{H}^n$ by $B(x,r) = \{y \in \mathbb{H}^n : d(x,y) < r\}$. Then, equipped with the Lebesgue measure on $\mathbb{R}^{2n+1}$, $\mathbb{H}^n$ is a space of homogeneous type.

7.4. The Caldéron–Zygmund integral on $\mathbb{H}^n$ is bounded on L^2

In $\mathbb{R}^N$, for a Caldéron–Zygmund kernel $K(z)$, we know that $f \longmapsto f * K$ is bounded on L^2. This result can be established using Plancherel's theorem. See also Theorem 7.5. Since K has mean value zero, it induces a distribution, hence it has a Fourier transform. Now K is homogeneous of degree $-N$, so we know $\widehat{K}$ is homogeneous of degree $-N-(-N) = 0$. Thus $\widehat{K}$ is bounded and

$$\|f * K\|_2 = \|\widehat{f * K}\|_2 = \|\widehat{f}\cdot\widehat{K}\|_2 \leq C\|\widehat{f}\|_2 = C\|f\|_2.$$

But we cannot use the same technique in $\mathbb{H}^n$ since we do not have the Fourier transform in $\mathbb{H}^n$ as a useful analytic tool. Instead we use the so-called Cotlar–Knapp–Stein lemma.

7.4.1. The Cotlar–Knapp–Stein lemma

Question. Let H be a Hilbert space. Suppose we have operators $T_j : H \to H$ that have uniformly bounded norm, $\|T_j\|_{\text{op}} = 1$. Then what can we say about $\|\sum_{j=1}^N T_j\|_{\text{op}}$?

It was Mischa Cotlar who first understood how to conceptualize this idea. Cotlar and Knapp/Stein independently found a much more flexible formulation of the result which has proved to be quite useful in the practice of harmonic analysis. We now formulate a version of their theorem (see [11,37]).

Lemma 7.6 (*Cotlar–Knapp–Stein*). *Let H be a Hilbert space and $T_j : H \to H$ be bounded operators, $j = 1, \ldots, N$. Suppose there exists a positive, bi-infinite sequence $\{a_j\}_{j=-\infty}^{\infty}$ of numbers such that $A = \sum_{-\infty}^{\infty} a_j < \infty$. Also assume that*

$$\|T_j T_k^*\|_{\text{op}} \le a_{j-k}^2, \qquad \|T_j^* T_k\|_{\text{op}} \le a_{j-k}^2. \tag{7.7.1}$$

Then

$$\left\| \sum_{j=1}^N T_j \right\|_{\text{op}} \le A.$$

7.4.2. The Folland–Stein theorem

Theorem 7.7 (*Folland, Stein, 1974*). *Let K be a function on $\mathbb{H}^n$ that is smooth away from 0 and homogeneous of degree $-2n-2$. Assume that*

$$\int_{|z|_h=1} K\,d\sigma = 0,$$

where $d\sigma$ is area measure (i.e., Hausdorff measure) on the unit sphere in the Heisenberg group. Define

$$Tf(z) = \text{PV}(K * f) = \lim_{\epsilon\to 0} \int_{|t|_h>\epsilon} K(t) f(t^{-1}z)\,dt.$$

Then the limit exists pointwise and in norm and

$$\|Tf\|_2 \le C\|f\|_2.$$

Remark. In fact, $T : L^p \to L^p$, for $1 < p < \infty$. We shall discuss the details of this assertion a bit later.

Theorem 7.1 is proved by breaking the integral up into dyadic pieces to which the Cotlar–Knapp–Stein theorem applies. We cannot provide the details here, but refer the reader instead to [70] or [48]. □

7.5. The Caldéron–Zygmund integral on $\mathbb{H}^n$ is bounded on L^p

In the last subsection we established L^2 boundedness of the Caldéron–Zygmund operators on the Heisenberg group. Given the logical development that we have seen thus far in the subject, the natural next step for us would be to prove a weak-type $(1, 1)$ estimate for these operators. Then one could apply the Marcinkiewicz interpolation theorem to get strong L^p estimates for $1 < p < 2$. Finally, a simple duality argument would yield strong L^p estimates for $2 < p < \infty$. And that would complete the picture for singular integrals on the Heisenberg group.

The fact is that a general paradigm for proving weak-type (1,1) estimates on a space of homogeneous type has been worked out in [9]. There are a number of interesting new twists and turns in this treatment – for instance the geometry connected with the Whitney decomposition is rather challenging – and we encourage readers to consult this original source as interest dictates. But it would be somewhat repetitious for us to present all the details here, and we shall not do so.

In the remainder of this paper, we shall take it for granted that L^p boundedness for Caldéron–Zygmund operators has been established, $1 < p < \infty$, and we shall use the result to good effect.

7.6. Applications of the Caldéron–Zygmund theorem

One fundamental result that can be derived from the preceding material concerns the mapping properties of the Szegő projection. Recall that the Szegő kernel $S(z, \zeta)$ is the canonical reproducing kernel for the space $H^2(\Omega)$. See [42] or [40] for details. And the mapping

$$L^2(\partial\Omega) \ni f \longmapsto \int_{\partial\Omega} f(\zeta)S(z, \zeta)\, d\sigma(\zeta)$$

is the Szegő projection. Obviously the Szegő projection is bounded on $L^2(\partial\Omega)$. But is it bounded on L^p?

It turns out, and this requires a substantial calculation to verify, that when Ω is the unit ball, then the Szegő kernel is a Caldéron–Zygmund kernel on the Heisenberg group as we have discussed above. Thus it is immediate that the Szegő projection maps L^p to L^p for $1 < p < \infty$. The Szegő projection comes up very naturally in other contexts of the function theory of several complex variables. See [8,43] for examples. The paper [57] contains generalizations to weakly pseudoconvex domains.

We shall treat some of these matters in the next section.

8. Analysis on the Heisenberg group

8.1. The Szegő kernel on the Siegel upper halfspace $\mathcal{U}$

Recall the height function (Section 7.3) ρ in $\mathcal{U}$:

$$\rho(w) = \operatorname{Im} w_1 - |w'|^2,$$

where $w' = (w_2, \ldots, w_n)$. We look at the almost analytic extension of ρ:

$$\rho(z, w) = \frac{i}{2}(\overline{w}_1 - z_1) - \sum_{k=2}^{n} z_k \overline{w}_k.$$

Note that ρ is holomorphic in z and conjugate holomorphic in w and $\rho(w, w) = \rho(w)$.

The next theorem is key to the principal result of this section. The Szegő kernel is a Heisenberg singular integral, hence can be analyzed using the machinery that we have developed.

Theorem 8.1. *On the Siegel upper half space $\mathcal{U}$, the Szegő kernel $S(z, \zeta)$ is:*

$$S(z, \zeta) = \frac{n!}{4\pi^n} \cdot \frac{1}{\rho(z, \zeta)^n}.$$

For $F \in H^2(\mathcal{U})$, we let

$$F_\rho(z', t) = F\big((z', t + i(|z'|^2 + \rho))\big).$$

We know that, for $z \in \mathcal{U}$,

$$F(z) = \int_{\partial\mathcal{U}} F_0(w) S(z, w) d\sigma(w).$$

This is just the standard reproducing property of the Szegő kernel acting on H^2 of the Siegel upper half space.

Corollary 8.2.

$$F_\rho(z', t) = F_0 * K_\rho(z', t),$$

where $F_0 \in L^2(\partial\mathcal{U})$ is the L^2 boundary limit of F and

$$K_\rho(z', t) = 2^{n+1} C_n \cdot (|z'|^2 + it + \rho)^{-n-1}, \quad C_n = \frac{n!}{4\pi^{n+1}}.$$

Theorem 8.3. *We have that*

$$S(z, w) = C_n \cdot [\rho(z, w)]^{-n-1},$$

where

$$\rho(z, w) = \frac{i}{2}(\overline{w}_1 - z_1) - \sum_{k=1}^{n} z_k \overline{w}_k$$

and

$$C_n = \frac{n!}{4\pi^{n+1}}.$$

Before we discuss the theorem we will formulate an important corollary. Since all our constructs are canonical, the Cauchy–Szegő representation ought to be modeled on a simple convolution operator on the Heisenberg group. Let us determine how to write the reproducing formula as a convolution.

A function F defined on $\mathcal{U}$ induces, for each value of the "height" ρ, a function on the Heisenberg group:

$$F_\rho(\zeta, t) = F\left(\zeta, t + i(|\zeta|^2 + \rho)\right).$$

Since $S(z, w)$ is the reproducing kernel, we know that

$$F(z) = \int_{\mathbb{H}^n} F(w)S(z, w)d\beta(w) \tag{8.3.1}$$

where $d\beta(w) = dw'du_1$ is the Haar measure on $\mathbb{H}^n$ with w written as $w = (u_1 + iv_1, w')$. Our discussion in Section 6 guarantees the existence of L^2 boundary values for F, and the boundary of $\mathcal{U}$ is $\mathbb{H}^n$. Thus the integral (8.3.1) is well-defined.

Observe that since the Heisenberg group is *not* commutative, we must be careful when discussing convolutions. We will deal with *right* convolutions, namely an integral of right translates of the given function F:

$$(F * K)(z) = \int_{\mathbb{H}^n} F(z \cdot y^{-1})K(y)dy = \int_{\mathbb{H}^n} F(y)K(y^{-1} \cdot z)dy.$$

The result we seek is

Corollary 8.4. *We have that*

$$F_\rho(\zeta, t) = F_0 * K_\rho(\zeta, t),$$

where F_0 is the L^2 boundary limit of F, and

$$K_\rho(\zeta, t) = 2^{n+1}c_n(|\zeta|^2 - it + \rho)^{-n-1}.$$

The completes our presentation of the main results of this section.

We now see the Szegő integral as a Heisenberg group convolution. An elementary analysis verifies that the kernel is in fact a Heisenberg group singular integral kernel. So we may invoke the Heisenberg Caldéron–Zygmund theorem to conclude that the Szegő projection is bounded on L^p, $1 < p < \infty$. It is *not* bounded on L^1 nor L^∞.

The paper [64] contains profound generalizations of Heisenberg group analysis to more general nilpotent Lie groups.

9. Reproducing kernels

9.1. Introduction

The Cauchy integral formula and the Poisson integral formula are perhaps the two most central and important examples of integral reproducing formulas. These are examples of *constructive* reproducing formulas (kernels) because the integral formulas (kernels) can often be written down explicitly or perhaps asymptotically (see [41,43,50]). What is of interest for our purpose here is that there are other integral reproducing formulas, which are canonical in nature, but for which the formulas (kernels) generally *cannot* be written down explicitly. Often the canonical kernels have many attractive features, but the fact that they are not explicit means that we do not necessarily understand their singularities, and therefore it is difficult to analyze them or to make estimates on them.

But there are techniques for making peace between the canonical and the constructive. The techniques presented here, due to Kerzman and Stein [36], are useful in other contexts.

9.2. Canonical integral formulas

In what follows, on $\mathbb{C}^n$, we use $d\sigma$ to represent area measure on a hypersurface.

Let Ω be a smoothly bounded domain in $\mathbb{C}$ or $\mathbb{C}^n$. Define

$$H^2(\Omega) = \left\{ f \text{ holomorphic on } \Omega : \sup_{0<\epsilon<\epsilon_0} \int_{\partial\Omega} |f(\zeta - \epsilon\nu_\zeta)|^2 \, d\sigma(\zeta)^{1/2} < \infty \right\},$$

where ν_ζ is the unit outward normal vector at the boundary point $\zeta \in \partial\Omega$. Clearly $H^2(\Omega)$ is a complex linear space. It is commonly called the *Hardy space*. We equip $H^2(\Omega)$ with the norm coming from its definition.

We have the following important preliminary result.

Lemma 9.1. *There is a constant $C = C(K, \Omega)$, depending only on the domain Ω and on K compact in Ω, such that, if $f \in H^2(\Omega)$, then*

$$\sup_{z\in K} |f(z)| \leq C \cdot \|f\|_{H^2(\Omega)}. \tag{9.1.1}$$

With the indicated norm, the space $H^2(\Omega)$ is a Hilbert space. First note that any element of H^2 may be canonically identified with an element of $L^2(\partial\Omega)$ (see [40, Ch. 8], for the details). For the completeness of $H^2(\Omega)$, note that if $\{f_j\}$ is a Cauchy sequence, then it will converge in the L^2 topology to some limit function g. But the lemma tells us that, for holomorphic functions, L^2 convergence implies uniform convergence on compact sets (sometimes called *normal convergence*). Hence the limit function is holomorphic and L^2, thus a member of H^2.

As a result of the lemma, the point evaluation

$$H^2(\Omega) \ni f \longmapsto f(z),$$

for z fixed in Ω, is a bounded linear functional on H^2. Let k_z be the Riesz representative of this functional.

We then define a function

$$S : \Omega \times \Omega \to \mathbb{C}$$

by the formula

$$S(z, \zeta) \equiv \overline{k_z(\zeta)}.$$

Of course S is easily extended (by Poisson integration) to $\Omega \times \Omega$. The function S is the *Szegő kernel* for the Hilbert space $H^2(\Omega)$.

We see that S is uniquely determined because, again by the Riesz representation theorem, the element k_z for each $z \in A^2$ is unique.

It is a classical fact, and we shall not provide the details here (but see [40] for all the particulars), that the kernel S may (at least in principle) be constructed by way of a complete orthonormal basis for H^2. To wit, let $\{\varphi_j\}$ be such a basis. Then

$$S(z,\zeta) = \sum_{j=1}^{\infty} \varphi_j(z)\overline{\varphi_j(\zeta)}.$$

Here the convergence is in the Hilbert space topology in each variables. And in fact the lemma shows that the convergence is uniform on compact subsets of $\Omega \times \Omega$.

We now need to present some preliminary results about the constructible Bochner–Martinelli kernel and integral representation formula.

Definition 9.2. On $\mathbb{C}^n$ we let

$$\begin{aligned}
\omega(z) &\equiv dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n \\
\eta(z) &\equiv \sum_{j=1}^{n} (-1)^{j+1} z_j dz_1 \wedge \cdots \wedge dz_{j-1} \wedge dz_{j+1} \wedge \cdots \wedge dz_n.
\end{aligned}$$

The form η is sometimes called the *Leray form.* We shall often write $\omega(\overline{z})$ to mean $d\overline{z}_1 \wedge \ldots \wedge d\overline{z}_n$ and likewise $\eta(\overline{z})$ to mean $\sum_{j=1}^{n}(-1)^{j+1}\overline{z}_j d\overline{z}_1 \wedge \ldots \wedge d\overline{z}_{j-1} \wedge d\overline{z}_{j+1} \wedge \ldots \wedge d\overline{z}_n$.

Theorem 9.3 (*Bochner–Martinelli*)**.** *Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain with C^1 boundary. Let $f \in C^1(\overline{\Omega})$. Then, for any $z \in \Omega$, we have*

$$\begin{aligned}
f(z) = {} & \frac{1}{nW(n)} \int_{\partial\Omega} \frac{f(\zeta)\eta(\overline{\zeta} - \overline{z}) \wedge \omega(\zeta)}{|\zeta - z|^{2n}} \\
& - \frac{1}{nW(n)} \int_{\Omega} \frac{\bar{\partial} f(\zeta)}{|\zeta - z|^{2n}} \wedge \eta(\overline{\zeta} - \overline{z}) \wedge \omega(\zeta).
\end{aligned}$$

Remark. We see that the Bochner–Martinelli formula is a quintessential example of a constructible integral formula. The kernel is quite explicit, and it is the same for all domains. For the Bergman kernel, and for other canonical kernels that we shall see below, this latter property does not hold. □

Corollary 9.4. *If $\Omega \subseteq \mathbb{C}^n$ is bounded and has C^1 boundary and if $f \in C^1(\overline{\Omega})$ and $\bar{\partial} f = 0$ on Ω, then*

$$f(z) = \frac{1}{nW(n)} \int_{\partial\Omega} \frac{f(\zeta)\eta(\overline{\zeta} - \overline{z})}{|\zeta - z|^{2n}} \wedge \omega(\zeta). \tag{9.4.1}$$

We note that the classical Cauchy integral formula in one complex variable is an immediate consequence of our new Bochner–Martinelli formula.

Corollary 9.5. *In complex dimension* 1 , *the last corollary says that*

$$f(z) = \frac{1}{2\pi i} \oint_{\partial\Omega} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

Corollary 9.4 is particularly interesting. Like the classical Cauchy formula, it gives a constructible integral reproducing formula that is the same on all domains. Unlike the classical Cauchy formula, its kernel is *not* holomorphic in the free variable z. This makes the Bochner–Martinelli formula of limited utility in *constructing* holomorphic functions.

We note that Corollary 9.4 holds for broader classes of holomorphic functions – such as the Hardy classes. One sees this by a simple limiting argument. See our discussion of H^2 below.

We conclude this section by noting that the integral

$$\mathbf{S}g(z) = \int_{\partial\Omega} S(z, \zeta) g(\zeta)\, d\zeta$$

defines a projection from $L^2(\partial\Omega)$ to $H^2(\Omega)$. This is because the mapping is self-adjoint, idempotent, and fixes H^2. We call this mapping the *Szegő projection*. [Note that the Bergman projection is constructed similarly.]

9.3. Constructive integral formulas with holomorphic kernel

In one complex variable it is easy to construct integral formulas. The Cauchy formula is quite trivial to write down. And it is the same for any domain. One may also write down formulas on the ball and polydisc in $\mathbb{C}^n$. After that things become complicated. Certainly one should mention here the classic work [32] in which the Bergman kernel is calculated for each of the Cartan bounded symmetric domains.

It was not until about 1970 that people found ways to write down integral reproducing formulas with holomorphic kernels in several complex variables. Here we discuss the idea. [It should be noted that both Bungart [4] and Gleason [24] proved some time ago – by abstract means – that reproducing kernels that are holomorphic in the free variable *always* exist. But the methods of [4,24] are nonconstructive, and the proofs quite abstract.]

Fix a non-negative integer k and a strongly pseudoconvex domain $\Omega \subset\subset \mathbb{C}^n$ with C^{k+3} boundary. Let $\rho : \mathbb{C}^n \to \mathbb{R}$ be a C^{k+3} defining function for Ω with the property that it is *strictly plurisubharmonic* in a neighborhood of $\partial\Omega$. It is a standard fact (see [40]), for which we do not provide the details, that the function (known as the *Levi polynomial*)

$$L : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$$

given by

$$\begin{aligned} L_P(z) = L(z, P) \equiv \rho(P) &+ \sum_{j=1}^{n} \frac{\partial\rho}{\partial z_j}(P)(z_j - P_j) \\ &+ \frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial^2\rho(P)}{\partial z_j \partial z_k}(z_j - P_j)(z_k - P_k) \end{aligned}$$

satisfies the following properties:

(9.3.1) For each $P \in \mathbb{C}^n$, the function $z \mapsto L(z, P)$ is holomorphic (indeed, it is a polynomial);
(9.3.2) For each $z \in \mathbb{C}^n$, the function $P \mapsto L(z, P)$ is C^{k+1};
(9.3.3) For each $P \in \partial\Omega$, there is a neighborhood U_P such that if $z \in \overline{\Omega} \cap \{w \in U_P : L_P(w) = 0\}$, then $z = P$.

Our goal is to remove the need to restrict to a small neighborhood of $P \in \partial\Omega$ (property (9.3.3)) while preserving properties (9.3.1)–(9.3.3). We proceed through a sequence of lemmas. Following Henkin, we use the notation

$$\Omega_\delta = \{z \in \mathbb{C}^n : \rho(z) < \delta\};$$
$$U_\delta = \{z \in \mathbb{C}^n : |\rho(z)| < \delta\}, \quad \delta > 0.$$

Further, let us fix the following constants:

(9.3.4) Choose $\delta > 0$ and $\gamma > 0$ such that

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(P) w_j \overline{w}_k \geq \gamma |w|^2, \quad \text{all } P \in U_\delta, \text{ all } w \in \mathbb{C}^n.$$

(9.3.5) Shrinking δ if necessary, we may select $\kappa > 0$ so that

$$|\mathrm{grad}\, \rho(z)| \geq \kappa \quad \text{for all } z \in U_\delta.$$

(9.3.6) With δ as above, let

$$K = \sum_{|\alpha|+|\beta|\leq 3} \left\| \left(\frac{\partial}{\partial z}\right)^{\alpha} \left(\frac{\partial}{\partial \overline{z}}\right)^{\beta} \rho(z) \right\|_{L^\infty(U_\delta)}.$$

Lemma 9.6. *There is a $\lambda > 0$ such that, if $P \in \partial\Omega$ and $|z - P| < \lambda$, then*

$$2\mathrm{Re}\, L_P(z) \leq \rho(z) - \gamma |z - P|^2/2.$$

Corollary 9.7. *Let $\epsilon = \gamma\lambda^2/20$. If $P \in \partial\Omega$, $z \in \Omega_\epsilon$, $\lambda/3 \leq |z - P| \leq 2\lambda/3$, then*

$$\mathrm{Re}\, L_P(z) < 0.$$

We may assume that $\epsilon < \lambda < \delta < 1$ (where δ is as in (9.3.4) and (9.3.5)). Let $\eta : \mathbb{R} \to [0, 1]$ be a C^∞ function that satisfies $\eta(x) = 0$ for $x \geq 2\lambda/3$ and $\eta(x) = 1$ for $x \leq \lambda/3$.

Lemma 9.8. *Fix $P \in \partial\Omega$. The $(0, 1)$ form*

$$f_P(z) = \begin{cases} -\bar{\partial}_z \{\eta(|z - P|)\} \cdot \log L_P(z) & \text{if } |z - P| < \lambda, z \in \Omega_\epsilon \\ 0 & \text{if } |z - P| \geq \lambda, z \in \Omega_\epsilon \end{cases}$$

is well-defined (if we take the principal branch for logarithm) and has C^∞ coefficients for $z \in \Omega_\epsilon$. If z is fixed, then $f_P(z)$ depends C^k on P. Finally, $\bar{\partial}_z f_P(z) = 0$ on Ω_ϵ. [One may note that this construction is valid even for P sufficiently near $\partial\Omega$.]

Lemma 9.9. *There is a C^∞ function u_P on Ω_ϵ such that $\bar\partial u_P = f_P$.*

This is just Hörmander's result – see Section 10.

We now define

$$\Phi(z,P) = \begin{cases} \left[\exp u_P(z)\right] \cdot L_P(z) & \text{if } |z-P| < \lambda/3 \\ \exp\left[u_P(z) + \eta(|z-P|)\log L_P(z)\right] & \text{if } \lambda/3 \le |z-P| < \lambda \\ \exp(u_P(z)) & \text{if } \lambda \le |z-P|. \end{cases}$$

Notice that Φ is unambiguously defined. To study the properties of Φ, we require two technical results.

Lemma 9.10. *If $U \subseteq \mathbb{C}^n$ is any open set and $K \subset\subset U$, then any $u \in C^1(U)$ satisfies*

$$\sup_K |u| \le C\left(\|u\|_{L^2(U)} + \|\bar\partial u\|_{L^\infty(U)}\right).$$

Here the constant C depends on U and K but not on u.

Corollary 9.11. *Let $\Omega \subset\subset \mathbb{C}^n$ be pseudoconvex and $K \subset\subset \Omega$. Let f be a $\bar\partial$-closed* (0, 1) *form with C^1 coefficients. If $u = Mf$ is the Hörmander solution to $\bar\partial u = f$ (see* [40, *Ch.* 4] *and Section* 10 *below), then we have*

$$\|u\|_{L^\infty(K)} \le C\|f\|_{L^\infty(\Omega)},$$

where C depends only on K and Ω (and not on f or u).

Proposition 9.12. *Assume once more that $\Omega \subset\subset \mathbb{C}^n$ has C^{k+3} boundary. Then $\Phi(\cdot, P)$ is holomorphic on Ω_ϵ. Also there is a $C > 0$, independent of P, such that for all $z \in \Omega_{\epsilon/2}$ we have*

(7.1) *if $|z-P| < \lambda/3$, then $|\Phi(z,P)| \ge C|L_P(z)|$;*

(7.2) *if $|z-P| \ge \lambda/3$, then $|\Phi(z,P)| \ge C$.*

Now we would like to consider the smooth dependence of Φ on P.

Fix $z \in \Omega$. Let $\theta \in C_c^\infty(\Omega_\epsilon)$ satisfy $\theta(z) = 1$. Let $s > 2n$. Let M_s^θ be the right inverse to $\bar\partial_{0,0}$ (the Hörmander solution operator) for the pseudoconvex domain Ω_ϵ. Let notation be as in (9.3.4) through (9.3.6). Let μ be in the dual space of $W^s(\Omega, \phi_1)$ [naturally this dual space is just $W^s(\Omega, \phi_1)$ itself]. Then

$$\langle \mu, M_s^\theta f_P\rangle = \langle (M_s^\theta)^*\mu, f_P\rangle,$$

which depends C^k on P because f_P does.

Proposition 9.13. *The function $\Phi(z,P)$ depends in a C^k fashion on P for fixed $z \in \Omega_\epsilon$.*

Proposition 9.14. *Let $\Omega \subseteq \mathbb{C}^n$ be pseudoconvex. Let $\Omega_m = \Omega \cap \{z \in \mathbb{C}^n : z_1, \ldots, z_m = 0\}$, $m = 1, \ldots, n$. Let $A_m(\Omega) = \{f$ holomorphic on $\Omega : f|_{\Omega_m} = 0\}$. Then there are linear operators*

$$Q_i^m : A_m(\Omega) \to \{f \text{ holomorphic on } \Omega\}, \quad i = 1, \ldots, m,$$

such that

$$f(z) = \sum_{i=1}^{m} z_i \cdot (Q_i^m f)(z)$$

for all $f \in A_m(\Omega)$.

Key in the result that we are about to present is the following extension result. See also Theorem 10.2 below.

Theorem 9.15. *Let* $\Omega \subseteq \mathbb{C}^n$ *be pseudoconvex (no assumptions about boundary smoothness, or even boundedness, need be made). Let* $\omega = \Omega \cap \{(z_1, \ldots, z_n) : z_n = 0\}$. *Let* $f : \omega \to \mathbb{C}$ *satisfy the property that the map*

$$(z_1, \ldots, z_{n-1}) \mapsto f(z_1, \ldots, z_{n-1}, 0)$$

is holomorphic on $\widetilde{\omega} = \{(z_1, \ldots, z_{n-1}) \in \mathbb{C}^{n-1} : (z_1, \ldots, z_{n-1}, 0) \in \omega\}$. *Then there is a holomorphic* $F : \Omega \to \mathbb{C}$ *such that* $F|_\omega = f$. *Indeed there is a linear operator*

$$\mathcal{E}_{\omega,\Omega} : \{\textit{holomorphic functions on } \omega\} \to \{\textit{holomorphic functions on } \Omega\}$$

such that $(\mathcal{E}_{\omega,\Omega} f)\big|_\omega = f$. *The operator is continuous in the topology of normal convergence.*

Corollary 9.16. *Let* $\Omega \subseteq \mathbb{C}^n$ *be pseudoconvex. Then there are continuous linear operators*

$$T_i : \{\textit{holomorphic functions on } \Omega\} \to \{\textit{holomorphic functions on } \Omega \times \Omega\}$$

such that, for any holomorphic $f : \Omega \to \mathbb{C}$, *we have*

$$f(z) - f(w) = \sum_{i=1}^{n} (z_i - w_i) T_i f(z, w), \quad \textit{all } z, w \in \Omega.$$

Proposition 9.17 (*Hefer's Lemma*)**.** *Let* $\Omega \subseteq \mathbb{C}^n$ *be strongly pseudoconvex with* C^4 *boundary. Let* $\Phi : \Omega_{\epsilon/2} \times \partial\Omega \to \mathbb{C}$ *be the* C^1 *singular function constructed above. Then we may write*

$$\Phi(z, \zeta) = \sum_{i=1}^{n} (\zeta_i - z_i) \cdot P_i(z, \zeta), \quad z \in \Omega_{\epsilon/2}, \quad \zeta \in \partial\Omega,$$

where each P_i *is holomorphic in* $z \in \Omega_{\epsilon/2}$ *and* C^1 *in* $\zeta \in \partial\Omega$.

We now quickly review the Cauchy–Fantappié formula. See [40, Ch. 5] for the details.

Theorem 9.18. *Let* $\Omega \subset\subset \mathbb{C}^n$ *be a domain with* C^1 *boundary. Let* $w(z, \zeta) = (w_1(z, \zeta), \ldots, w_n(z, \zeta))$ *be a* C^1, *vector-valued function on* $\bar{\Omega} \times \bar{\Omega} \setminus \{\textit{diagonal}\}$ *that satisfies*

$$\sum_{j=1}^{n} w_j(z, \zeta)(\zeta_j - z_j) \equiv 1.$$

Then, using the notation from Section 9.2, we have for any

$$f \in C^1(\bar{\Omega}) \cap \{\textit{holomorphic functions on } \Omega\}$$

and any $z \in \Omega$ the formula

$$f(z) = \frac{1}{nW(n)} \int_{\partial\Omega} f(\zeta)\eta(w) \wedge \omega(\zeta).$$

We see that the Cauchy–Fantappié formula is a direct generalization of the Bochner–Martinelli formula discussed above. Now we can give the punchline of this development.

Theorem 9.19 (*Henkin [28]*). *Let $\Omega \subseteq \mathbb{C}^n$ be a strongly pseudoconvex domain with C^4 boundary. Let $\Phi : \Omega_{\epsilon/2} \times \partial\Omega \to \mathbb{C}$ be the Henkin singular function. Define*

$$w_i(z,\zeta) = \frac{P_i(z,\zeta)}{\Phi(z,\zeta)}, \quad i = 1, \ldots, n.$$

Here $P_i(z,\zeta)$ are as in Proposition 9.17. Just as in our earlier discussion, let

$$\eta(w) = \sum_{i=1}^{n} (-1)^{i+1} w_i dw_1 \wedge \cdots \wedge dw_{i-1} \wedge dw_{i+1} \wedge \cdots \wedge dw_n$$

and

$$\omega(\zeta) = d\zeta_1 \wedge \cdots \wedge d\zeta_n.$$

Then, for any $f \in C^1(\overline{\Omega}) \cap \{$holomorphic functions on $\Omega\}$, we have the integral representation

$$f(z) = \int_{\partial\Omega} f(\zeta)\eta(w) \wedge \omega(\zeta).$$

Corollary 9.20. *With notation as in the theorem, we have*

$$f(z) = \int_{\partial\Omega} f(\zeta) \frac{K(z,\zeta)}{\Phi^n(z,\zeta)} d\sigma(\zeta), \tag{9.20.1}$$

where $K : \Omega_{\epsilon/2} \times \partial\Omega$ is holomorphic in z and continuous in ζ. In fact, $K(z,\zeta)d\sigma(\zeta) = \eta(z) \wedge \omega(\zeta)$.

Of course Corollary 9.20 gives us a constructive integral reproducing formula with kernel that is holomorphic in the free z variable. This is a very useful device, and important for the function theory of several complex variables.

9.4. Asymptotic expansion for the canonical kernel

C. Fefferman [16] made an important contribution in 1974 when he produced an asymptotic expansion for the Bergman kernel of a strongly pseudoconvex domain. Basically he was able to write

$$K(z,\zeta) = P(z,\zeta) + \mathcal{E}(z,\zeta),$$

where P (the principal term) is, in suitable local coordinates, the Bergman kernel of the ball and $\mathcal{E}$ (the error term) is a term of strictly lower order (in some measurable sense). This powerful formula gives one a means for calculating mapping properties of the Bergman integral. Fefferman himself used the formula to calculate the boundary asymptotics of Bergman metric geodesics (for the purpose of proving the smooth boundary extension of biholomorphic mappings). Fefferman states in his paper – although the details have never been worked out – that there is a similar asymptotic expansion for the Szegő kernel of a strongly pseudoconvex domain.

At about the same time, Boutet de Monvel and Sjöstrand [3] used the technique of Fourier integral operators [31] to directly derive an asymptotic expansion for the Szegő kernel of a strongly pseudoconvex domain. This expansion is quite similar to Fefferman's: there is a principal term, which in suitable local coordinates is the Szegő kernel of the ball, and there is an error term which is of lower order. Theorem 1.5 of [3] treats means of applying these Fourier integral operator techniques to the study of the Bergman kernel on strongly pseudoconvex domains.

The main purpose of the present discussion is to consider another method, due to Kerzman and Stein, for deriving asymptotic expansions for the canonical kernels that is more elementary and uses less machinery. At this time there are virtually no results about asymptotic expansions for the canonical kernels on weakly pseudoconvex domains. Some interesting partial results appear in [27]. See also [49].

9.5. The relation between constructive kernels and canonical kernels on strongly pseudoconvex domains

In previous sections, we have defined the Szegő projection $\mathbf{S} : L^2(\partial\Omega) \to H^2(\Omega)$. We also have a mapping $\mathbf{H} : L^2(\partial\Omega) \to H^2(\Omega)$ that is determined by the Henkin kernel of Corollary 9.20. We note that $\mathbf{H}$ defines a bounded operator from $L^2(\partial\Omega)$ to $H^2(\Omega)$ (the Hardy space – see [40, Chapter 8]) for the following reason.

It is known that $\partial\Omega$, when equipped with balls coming from the complex structure and the usual boundary area measure (see [59,60]), is a space of homogeneous type in the sense of Coifman and Weiss [9]. Further, it is straightforward to verify that the Henkin operator $\mathbf{H}$ satisfies the hypotheses of the David–Journé $T1$ theorem for spaces of homogeneous type (see [7] for a nice exposition of these ideas). Thus we may conclude that the Henkin operator maps $L^2(\partial\Omega)$ to $L^2(\partial\Omega)$. Since the Henkin kernel also obviously maps $L^2(\partial\Omega)$ to holomorphic functions, we may conclude that the Henkin integral maps $L^2(\partial\Omega)$ to $H^2(\Omega)$. This mapping, however, is *not* a projection.

Now of course $\mathbf{S}$, being a projection, is self-adjoint. So $\mathbf{S} = \mathbf{S}^*$. It is not at all true that $\mathbf{H} = \mathbf{H}^*$, but one may calculate that $\mathcal{A} \equiv \mathbf{H}^* - \mathbf{H}$ is small in a measurable sense.

We also have

$$\begin{aligned} \mathbf{HS} &= \mathbf{S}, \qquad \mathbf{SH}^* = \mathbf{S}, \\ \mathbf{SH} &= \mathbf{H}, \qquad \mathbf{H}^*\mathbf{S} = \mathbf{H}^*. \end{aligned}$$

Let us discuss these four identities for a moment.

For the first, notice that $\mathbf{S}$ is the projection onto H^2, and $\mathbf{H}$ preserves holomorphic functions. So certainly $\mathbf{HS} = \mathbf{S}$. For the second, we calculate that

$$\langle \mathbf{SH}^* x, y\rangle = \langle \mathbf{H}^* x, \mathbf{S}y\rangle = \langle x, \mathbf{HS}y\rangle = \langle x, \mathbf{S}y\rangle$$

(because $\mathbf{H}$ preserves holomorphic functions) and thus $= \langle \mathbf{S}x, y\rangle$. Hence $\mathbf{SH}^* = \mathbf{S}$. For the third, notice that $\mathbf{H}$ maps to the holomorphic functions, and $\mathbf{S}$ preserves holomorphic functions. And, for the fourth, we calculate that

$$\langle \mathbf{H}^*\mathbf{S}x, y\rangle = \langle \mathbf{S}x, \mathbf{H}y\rangle = \langle x, \mathbf{SH}y\rangle = \langle x, \mathbf{H}y\rangle = \langle \mathbf{H}^*x, y\rangle.$$

In conclusion, $\mathbf{H}^*\mathbf{S} = \mathbf{H}^*$.

Now we see that

$$\mathbf{S}\mathcal{A} = \mathbf{S}(\mathbf{H}^* - \mathbf{H}) = \mathbf{SH}^* - \mathbf{SH} = \mathbf{S} - \mathbf{H}.$$

As a result,

$$\mathbf{S} = \mathbf{H} + \mathbf{S}\mathcal{A}$$

so

$$\mathbf{S}(\mathbf{I} - \mathcal{A}) = \mathbf{H}.$$

In conclusion,

$$\mathbf{S} = \mathbf{H}(\mathbf{I} - \mathcal{A})^{-1}.$$

If indeed we can show that $\mathcal{A}$ is norm small in a suitable sense, then $(\mathbf{I} - \mathcal{A})^{-1}$ is well defined by a Neumann series. Thus we may write

$$\mathbf{S} = \mathbf{H} + \mathbf{H}\mathcal{A} + \mathbf{H}\mathcal{A}^2 + \cdots + \mathbf{H}\mathcal{A}^j + \mathbf{H}\mathcal{A}^{j+1} + \cdots.$$

Hence we have expressed the Szegő projection $\mathbf{S}$ as an asymptotic expansion in terms of the Henkin projection $\mathbf{H}$. By applying this asymptotic expansion to the Dirac delta mass, this last formula can be translated into saying that the Szegő *kernel* S can be written as an asymptotic expansion in terms of the Henkin *kernel*.

It should be noted that Ewa Ligocka [54] has shown that these same ideas may be applied to expand the Bergman kernel in an asymptotic expansion in terms of the Henkin kernel. We shall not treat the details of her argument here.

10. The inhomogeneous Cauchy–Riemann equations

In the function theory of one complex variables we study the Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

For a C^1 function $h = u + iv$, these equations give a characterization of holomorphicity. They are a powerful tool in elementary function theory.

In several complex variables it is useful to introduce the partial differential operators

$$\frac{\partial}{\partial z_j} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \quad j = 1, 2, \ldots, n$$

and

$$\frac{\partial}{\partial \overline{z}_j} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right), \quad j = 1, 2, \ldots, n.$$

We say that a C^1 function f of n complex variables is *holomorphic* if $\partial f/\partial \overline{z}_j \equiv 0$ for $j = 1, \ldots, n$. This is just a new notation for saying that f satisfies the classical Cauchy–Riemann equations in each complex variable separately.

An innovation in several complex variables is that we also study the *inhomogeneous* Cauchy–Riemann equations. These are a powerful device for constructing holomorphic functions. To understand this tool, we define for a C^1 function u the operator

$$\overline{\partial} u \equiv \sum_{j=1}^{n} \frac{\partial u}{\partial \overline{z}_j}\, d\overline{z}_j.$$

Thus $\overline{\partial}$ is an operator from functions to (0, 1) forms. We also define

$$\overline{\partial}\left(\sum_{j=1}^{n} \alpha_j(z) d\overline{z}_j\right) = \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial \alpha_j}{\partial \overline{z}_k} d\overline{z}_k \wedge d\overline{z}_j.$$

Clearly $\overline{\partial}\overline{\partial} = 0$.

Thus it makes sense to study the partial differential equation

$$\overline{\partial} u = f, \tag{10.1}$$

where f is a given, $\overline{\partial}$-closed (0, 1) form and u is an unknown function. [One can also study this PDE in one complex variables. But, because of issues connected with the support of the solution, this is a much less useful tool in that context – see [40, Ch. 1] for the details].

Here is a standard and universally applicable result on the $\overline{\partial}$ problem. The reference is [31].

Theorem 10.1. *Let $\Omega \subseteq \mathbb{C}^n$ be a bounded, pseudoconvex domain. Let f be a $\overline{\partial}$-closed (0, 1) form on Ω with L^2 coefficients. Then there is an L^2 function u on Ω such that $\overline{\partial} u = f$. We have the estimate*

$$\|u\|_{L^2} \leq C\|f\|_{L^2}.$$

The constant C depends only on the diameter of the domain. If f has C^∞ coefficients then so does u.

Hörmander's theorem is in fact true in much greater generality. In that broader context, the estimates are formulated in terms of a plurisubharmonic weight function.

This theorem has been profoundly influential in many fields, including algebraic and complex geometry. A nice exposition of the latter ideas appears in [12].

A typical application of the partial differential equation (10.1) is as follows.

Theorem 10.2. *Let $\o \subseteq \mathbb{C}^n$ be pseudoconvex with C^2 boundary. Let $\omega = \o \cap \{(z_1, \ldots, z_n) : z_n = 0\}$. Let $f : \omega \to \mathbb{C}$ satisfy the property that the map*

$$(z_1, \ldots, z_{n-1}) \mapsto f(z_1, \ldots, z_{n-1}, 0)$$

is holomorphic on $\tilde{\omega} = \{(z_1, \ldots, z_{n-1}) \in \mathbb{C}^{n-1} : (z_1, \ldots, z_{n-1}, 0) \in \omega\}$. *Then there is a holomorphic* $F : \Omega \to \mathbb{C}$ *such that* $F|_\omega = f$. *Indeed there is an operator*

$$\mathcal{E}_{\omega,\emptyset} : \{\textit{holomorphic functions on } \omega\} \to \{\textit{holomorphic functions on } \emptyset\}$$

such that $\left.(\mathcal{E}_{\omega,\emptyset} f)\right|_\omega = f$. *The operator is continuous in the topology of normal convergence.*

Proof. Let $\pi : \mathbb{C}^n \to \mathbb{C}^n$ be the Euclidean projection $(z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_{n-1}, 0)$. Let $\mathcal{B} = \{z \in \emptyset : \pi z \notin \omega\}$. Then $\mathcal{B}$ and ω are *relatively closed* disjoint subsets of $\emptyset$. Hence there is a function $\Psi : \emptyset \to [0, 1]$, $\Psi \in C^\infty(\emptyset)$, such that $\Psi \equiv 1$ on a relative neighborhood of ω and $\Psi \equiv 0$ on $\mathcal{B}$. [This last assertion is intuitively non-obvious. It is a version of the C^∞ Urysohn lemma, for which see [29]. It is also a good exercise for the reader to construct Ψ by hand]. Set

$$F(z) = \Psi(z) \cdot f(\pi(z)) + z_n \cdot v(z),$$

where v is an unknown function to be determined.

Notice that $f(\pi(z))$ is well-defined on supp Ψ. We wish to select $v \in C^\infty(\emptyset)$ so that $\bar{\partial} F = 0$. Then the function F defined by the displayed equation will be the function that we seek.

Thus we require that

$$\bar{\partial} v(z) = \frac{\left(-\bar{\partial}\,\Psi(z)\right) \cdot f\left(\pi(z)\right)}{z_n}. \tag{10.2.1}$$

Now the right side of this equation is C^∞ since $\bar{\partial}\,\Psi \equiv 0$ on a neighborhood of ω. Also, by inspection, the right side is annihilated by the $\bar{\partial}$ operator (remember that $\bar{\partial}^2 = 0$). By Theorem 10.1, there exists a $v \in C^\infty(\emptyset)$ that satisfies (10.2.1). Therefore the extension F exists and is holomorphic.

Following the proof of Theorem 10.1, we may check that the operator $\mathcal{E}_{\omega,\Omega}$ is bounded in L^2 of any compact set. But then the Cauchy estimates show that the operator is bounded in the topology of normal convergence. $\square$

Remark 10.3. An obverse of Theorem 10.2 is the noted Ohsawa–Takegoshi extension theorem (see [61]). This is a widely used result, with notable applications in algebraic geometry.

It is a matter of some interest to determine estimates for the Eq. (10.1). That is, if the coefficients of f lie in a particular Banach space, then what can we say about the attributes of u? In what Banach space will u lie?

Studies of strongly elliptic partial differential equations (see [44]) lead one to think that, if f has coefficients in W^r (the standard Sobolev space), then u should show an improvement in smoothness corresponding to the degree of the partial differential operator. This is what happens, for instance, with the Laplacian. But the $\overline{\partial}$ operator is *not* elliptic up to the boundary. Instead it is *subelliptic*. So it has a less-than-optimal regularity theory.

The earliest estimates for the $\overline{\partial}$ problem were due to J.J. Kohn (see [18] and the references therein). He showed that, when Ω is a smoothly bounded, strongly pseudoconvex domain, then

$$\|u\|_{W^{s+1/2}} \leq C\|f\|_{W^s}$$

so long as u is the so-called canonical solution to the $\overline{\partial}$ problem. Here the canonical solution is that which is orthogonal to holomorphic functions; it is given by $u_c = u - Pu$, where P is the Bergman projection.

A few years later, Lars Hörmander [31] proved that the equation $\bar{\partial}u = f$ has a solution u that satisfies

$$\|u\|_{L^2} \leq C\|f\|_{L^2}$$

on *any* bounded pseudoconvex domain.

It was considered a breakthrough when, around 1970, Grauert/Lieb [25] and Kerzman [35] and Ramirez [63] and Henkin [28] independently proved uniform estimates for the $\bar{\partial}$ problem. More precisely, they showed that, on a strongly pseudoconvex domain Ω, a particular solution (which can be explicitly constructed) of the equation $\bar{\partial}u = f$ satisfies

$$\|u\|_{L^\infty} \leq C\|f\|_{L^\infty}.$$

Kerzman also proved L^p estimates.

Later, Siu [67] built on this work and showed that

$$\|u\|_{C^k} \leq C_k\|f\|_{C^k}$$

for every k on a smoothly bounded, strongly pseudoconvex domain.

In his 1974 Ph.D. thesis, Krantz [45] obtained sharp estimates for the $\bar{\partial}$ problem in strongly pseudoconvex domains. In particular, he showed that, for an explicitly constructible solution (due to Henkin) j of the $\overline{\partial}$ problem,

$$\|u\|_{L^q} \leq C \cdot \|f\|_{L^p}$$

for $1 < p < 2n + 2$ and $\frac{1}{q} = \frac{1}{p} - \frac{1}{2n+2}$. Also

$$\|u\|_{\mathrm{Lip}_c} \leq C \cdot \|f\|_{L^p}$$

for $2n + 2 < p \leq \infty$ and $c = \frac{1}{2} - \frac{n+1}{p}$.

Krantz also proved estimates in various nonisotropic Lipschitz norms, but we shall not detail the results here. See [45] for the particulars.

It is certainly possible to combine the results of Siu [67] and Krantz [45] to obtain sharp estimates on the derivatives of the Henkin solution u of the equation $\bar{\partial}u = f$. We shall not provide the details here.

There are further developments in the realm of estimates for the $\bar{\partial}$ problem. As an example, Greiner and Stein [26] proved sharp estimates on strongly pseudoconvex domains for $\bar{\partial}$ for the *canonical solution* (the one that is orthogonal to holomorphic functions). This is profound work, and ties together many of the earlier ideas that we have described above.

11. Ideas of Christ/Geller

One of the beautiful features of harmonic analysis on $\mathbb{R}^n$ is that we have nice characterizations of the real-variable H^p spaces in terms of the Riesz transforms – see [74,17], for instance. It is not at all obvious how to develop an analogous theory on the ball in $\mathbb{C}^n$. But Christ and Geller [8] made great strides in that direction. We should like to describe some of their ideas here.

First we note that, in the pioneering work of Stein and Weiss [74], the real variable Hardy spaces were *defined* in terms of the Riesz transforms. Just as, on the real line, a function f is in H^1 if and only if $f \in L^1$ and the Hilbert transform Hf is in L^1, so it is that a function on $\mathbb{R}^N$ is in real-variable H^1 if and only if $f \in L^1$ and the Riesz transforms $R_j f \in L^1$, where

$$R_j f(x) \equiv \text{P.V.} \int_{\mathbb{R}^N} f(t) \cdot \frac{x_j - t_j}{|x-t|^{N+1}} \, dt.$$

A natural question to ask at this point is which other collections of N singular integrals will characterize H^1. In a pioneering work [76], Akihito Uchiyama gave us the answer. Consider the convolution operators

$$K_j f(x) \equiv \left(\theta_j(\xi/|\xi|)\widehat{f}(\xi)\right)^{\vee}(x),$$

with θ_j in C^∞ of the unit sphere and satisfying the condition

$$\text{rank} \begin{pmatrix} \theta_1(\xi) \cdots \theta_m(\xi) \\ \theta_1(-\xi) \cdots \theta_m(-\xi) \end{pmatrix} \equiv 2 \quad \text{for } |\xi| = 1.$$

Such a system of singular integrals K_j will in fact characterize $H^1_{\text{Re}}(\mathbb{R}^N)$. [It may be noted that the Riesz transform kernels obviously satisfy this condition.]

Now Christ and Geller [8] built on the ideas of Uchiyama to prove the following result. It is the first of its kind along the lines of characterizing H^p on a homogeneous group using singular integrals. Prior to the work in [8], the only way to approach Hardy spaces on homogeneous groups was by way of the atomic theory – see [9].

Now the result of [8] is this: A collection K_j of singular integral operators on the Heisenberg group characterizes (in the sense discussed in the preceding paragraph) H^1 of the Heisenberg group if, for each $v \in \mathbb{R}^m$, there are singular integral operators L_j such that $\sum K_j * L_j = I$ and $\sum v_j L_j = 0$. We cannot provide here all the technical details of the Christ/Geller argument. Let us just say that this is a foundational result for harmonic analysis on the unit ball in $\mathbb{C}^n$, and much work remains to be done in developing this point of view.

The entire idea of developing a working harmonic analysis on the Heisenberg group was explored by D. Geller in [20–22]. See also [23].

12. Square functions

The idea of square functions goes back at least to work of Lusin. We learn in a basic function theory course that, if $U \subseteq \mathbb{C}$ is a bounded domain and $f : U \to \mathbb{C}$ is a univalent holomorphic mapping, then $f(U)$ is a domain and

$$\int_U |f'(\zeta)|^2 \, dA(\zeta) = \int_{f(U)} 1 \, dA(\zeta) = \text{area}\,(f(U)).$$

This is simply the change of variables formula, for $|f'(\zeta)|^2$ is nothing other than the Jacobian determinant of the mapping.

Thanks to work of E.M. Stein and others, the idea of the square function has been generalized to the harmonic analysis of several variables. And, more recently, Stein [69] has proved results about the square function on strongly pseudoconvex domains. Following that work, Krantz and Li [51,52] established results about the square function on finite type domains in $\mathbb{C}^n$.

A detailed history of the square function is given in [73]. Its history began with a theorem of Kaczmarz and Zygmund in 1926 (see [33,79]). It slowly evolved, through work of Caldéron, Zygmund, and others, to a more modern form that we shall emphasize here.

We work on the upper halfspace $\mathbb{R}^{N+1}_+ \equiv \{x = (x_1, x_2, \ldots, x_N, t) : t > 0\}$. If $x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N = \partial\mathbb{R}^{N+1}_+$, then let

$$\Gamma(x) = \{(x, t) : |x| < t\}.$$

For a harmonic function u on the upper halfspace $\mathbb{R}^{N+1}_+$ which is the Poisson integral of an integrable function f on the boundary $\mathbb{R}^N$, and for $x \in \partial\mathbb{R}^{N+1}_+$, we define

$$[Au(x)]^2 = \int_{\Gamma(x)} |\nabla u|^2 y^{1-N}\, dxdy.$$

Notice that this is an $(N+1)$-dimensional integral. The presence of the factor y^{1-N} is justified by dimensionality considerations (it trivializes to the zeroth power in the classical setting of the disc). It will be put into a more natural context when we formulate the area integral on a strongly pseudoconvex domain in the language of the Bergman metric.

Now the fundamental theorem about the operator A is as follows:

Theorem 12.1. *For* $1 < p < \infty$,

$$\|Au\|_{L^p} \cong \|f\|_{L^p}.$$

In this formulation, the result is due to Stein [71]. Stein's original rather complicated proof has been simplified (see, for instance [70]). But we still cannot present the details here.

Theorem 12.2. *Let Ω be a smoothly bounded, strongly pseudoconvex domain in $\mathbb{C}^n$. Let F be a holomorphic function on Ω. Then, for almost every $\zeta \in \partial\Omega$, the following are equivalent:*

1. *F is admissibly bounded at ζ;*
2. *F has an admissible limit at ζ;*
3. $$\int_{\mathcal{A}_\alpha(\zeta)} |\nabla F|^2\, dM(z) < \infty.$$

Here it should be understood that ∇F is calculated in the metric.

Stein's proof is rather technical, and we cannot reproduce it here. In some sense the argument is a sophisticated application of Stokes' theorem.

In later work, Krantz and Li [51,52] were able to generalize what Stein did in [69]. For a fixed domain Ω, $z \in \Omega$, and $\alpha > 0$, define

$$D_\alpha(z) = \{w \in \Omega : \pi(w) \in B(z, \alpha\delta(w))\}.$$

Also let

$$d\lambda(z) = K(z, z) dv(z),$$

where K is the Bergman kernel and dv is Euclidean volume measure. Now we define the *area integral* of a function f on Ω to be

$$A_\alpha f(z) = \left(\int_{D_\alpha(z)} |\nabla f(w)|^2 \delta(w)^2 \, d\lambda(w) \right)^{1/2} .$$

The maximal function of a given function f on Ω is given by

$$f_\alpha^*(z) = \sup \{|f(w)| : w \in D_\alpha(a)\} .$$

The radial maximal function is defined to be

$$f^+(z) = \sup \{|f(z + t\nu(z))| : 0 \le t \le \epsilon_0\} .$$

The Littlewood–Paley g-function is given by

$$g(f)(z) = \left[\int_0^{\epsilon_0} |\nabla f(z + t\nu(z))|^2 t \, dt \right]^{1/2} .$$

Now we have the following theorem:

Theorem 12.3. *Let Ω be a smoothly bounded domain which is either strongly pseudoconvex, of finite type in $\mathbb{C}^2$, or convex and of finite type in any dimension. Let $0 < p < \infty$ and let Ω be a regular domain in $\mathbb{C}^n$. Then the following are equivalent:*

1. $f \in \mathcal{H}^p(\Omega)$.
2. $f^+ \in L^p(\partial\Omega)$.
3. $f_\alpha^* \in L^p(\partial\Omega)$.
4. $g(f) \in L^p(\partial\Omega)$.
5. $A_\alpha(f) \in L^p(\partial\Omega)$.

Of course the ideas here originate in [62]. Their modern multivariable form was initiated in [69].

13. Ideas of Nagel/Stein and Di Biase

As mentioned earlier, the traditional wisdom has been that, in the context of harmonic functions on the disc, the optimal approach regions for boundary limits are the nontangential approach regions. Well established examples of Littlewood, Rudin, and others reinforce this notion. Thus it was a remarkable observation of Nagel and Stein [58] that what is important about the shape of the approach region is *not* its shape (in the common sense of the word “shape”), but rather the measure of its cross section. That is, it turns out that what is important about the nontangential approach region

$$\Gamma_\alpha(P) = \{z \in D : |z - P| < \alpha(1 - |z|)\}$$

is that

The one-dimensional measure of the set of points in $\Gamma_\alpha(P)$ that are distance δ from P is about δ. (*)

It is not difficult to see that one can construct a region having property (*) that is *not* nontangential. Yet, through such a region, there are boundary limits for p^{th} power integrable harmonic functions. The technique of Nagel and Stein involves a new version of the Hardy–Littlewood maximal function.

Fausto Di Biase – see for instance [14,13] – was able to generalize the ideas of Nagel and Stein to the several complex variable setting. Thus he replaces admissible approach regions by regions that have cross sections with the same area as $\mathcal{A}_\alpha$. And then there are boundary limits for H^p functions through these new approach regions. Di Biase’s work is rather complex, involving harmonic analysis on trees.

14. Concluding remarks

There are many aspects of analysis and several complex variables that we have not considered here. Just as an instance, the Szegő and Bergman kernels have had a powerful impact on Kähler geometry (by way of the recently proved Yau–Tian–Donaldson conjecture, just as an instance). The book [55] gives a glimpse of some of these developments.

The theory of harmonic analysis in the several complex variables setting is really quite young. We have only begun to scratch the surface of what is possible. In particular, while the strongly pseudoconvex situation is fairly well understood, that for finite type domains or even more general domains is mostly an open book. New techniques are required in order to make any meaningful progress.

It is clear that the harmonic analysis of several complex variables will be key to understanding the corona problem and other important problems in the function theory of several complex variables. This is a field in which it is worth investing some effort.

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